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A REFINED ASYMPTOTIC THEORY FOR DYNAMIC ANALYSIS OF DOUBLY CURVED LAMINATED SHELLS

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Abstract—The asymptotic theory developed recently for dynamic analysis of doubly curved laminated shells is refined by including the transverse rotations as auxiliary variables. The theory embraces the first-order shear deformation theory (FSDT) and the higher-order shear deformation theory (HSDT) as the first-order approximation. Higher-order corrections to the approximation are determined by solving the FSDT or HSDT equations in a hierarchic way. The secular terms in the asymptotic solution are eliminated systematically by means of multiple scales and solvability conditions for the higher-order equations. The performance of the refined theory is illustrated by applying it to benchmark problems. Numerical comparisons are made to examine the convergence of the solutions. © 1998 Elsevier Science Ltd.

1. INTRODUCTION

In analyzing the dynamic responses of laminated shells, various displacement models have been proposed by assuming *a priori* the variation of the displacements through the thickness, such as the classical laminated shell theory (CST) (Flügge, 1973; Leissa and Qatu, 1991), the first-order shear deformation theory (FSDT) and the higher-order shear deformation theory (HSDT) (Reddy, 1984; Reddy and Liu, 1985; Librescu *et al.*, 1989; Khdeir *et al.*, 1989). These models provide a simple way for predicting the structural responses of the laminated system, yet they are generally inadequate in considering the heterogeneity effect through the thickness. In two recent papers (Wu *et al.*, 1996a, b), an asymptotic theory of multilayer double-curved shells has been presented. The essential feature is that a three-dimensional solution to the problem can be determined by solving the CST equations hierarchically. There is no need to treat the system layer by layer, nor to consider the interfacial continuity conditions in particular; the continuity requirements are satisfied inherently in the formulation.

It is well recognized that the effects of transverse shear deformations are significant in laminated shells, especially in case there is a large difference in the layer properties. Under these circumstances, the CST is unreliable since the transverse shear deformations are completely neglected. In the asymptotic formulation presented earlier the transverse shear deformations enter the picture only after the leading order, resulting in CST equations as the governing equations at each level of approximation. Consequently, the convergence of the asymptotic solution is slow for problems of thick shells. As the higher-order displacement models usually yield better results than the CST does, it is desirable to have the asymptotic theory refined in such a way that the FSDT or the HSDT equations become the governing equations in order to accelerate the convergence, but not at the expense of reformulation. With this in mind, we extend the asymptotic theory to embrace FSDT and HSDT as the first-order approximation. The basic idea is to introduce the shear rotations in the formulation as auxiliary variables by bringing the transverse shear deformations to the stage right from the leading-order level. The asymptotic integration leads naturally to a refined theory with the shear deformation theory as the first-order approximation. Higher-order corrections can be determined by solving the FSDT or HSDT equations hierarchically. As before, the secular terms in the expansions are eliminated in a systematic way by using the method of multiple scales (Nayfeh, 1981) and considering the solvability

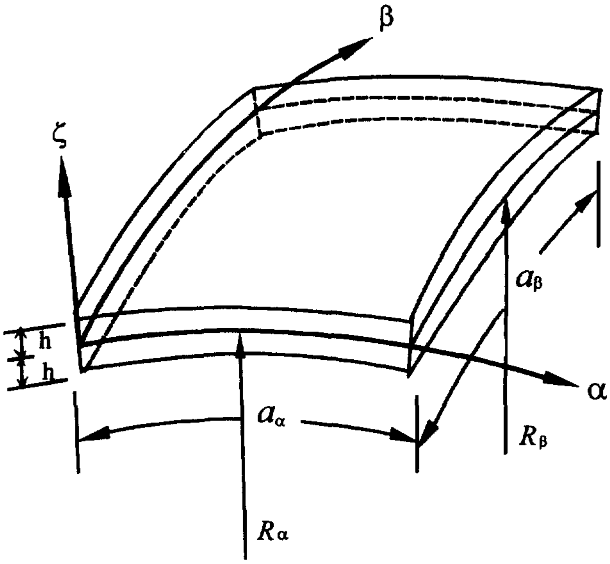

Fig. 1. The geometry and coordinate system for the doubly curved shell.

conditions of the higher-order equations. The performance of the refined theory is examined by applying it to benchmark problems. The solution convergence is improved significantly over the one based on the asymptotic theory without refinement.

2. BASIC FORMULATION

2.1. *Three-dimensional equations*

Consider an anisotropic, heterogeneous doubly curved shell as shown in Fig. 1. The two radii of curvature are assumed to be constant independent of position. The thickness of the shell is uniform and denoted by $2h$. The orthogonal curvilinear coordinates α, β, ζ are located on the middle surface. R_α and R_β denote the radii of curvature to the middle surface, a_α and a_β are the curvilinear dimensions in α and β directions, respectively. The materials are curvilinear monoclinic having at each point elastic symmetry with respect to the surfaces ζ = constant.

The stress–strain relations for the curvilinear monoclinic materials expressed relative to the geometrical axes are

$$\begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\zeta \\ \tau_{\beta\zeta} \\ \tau_{\alpha\zeta} \\ \tau_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\zeta \\ \gamma_{\beta\zeta} \\ \gamma_{\alpha\zeta} \\ \gamma_{\alpha\beta} \end{Bmatrix} \tag{1}$$

where σ_α, σ_β, σ_ζ, $\tau_{\alpha\zeta}$, $\tau_{\beta\zeta}$, $\tau_{\alpha\beta}$ and ε_α, ε_β, ε_ζ, $\gamma_{\alpha\zeta}$, $\gamma_{\beta\zeta}$, $\gamma_{\alpha\beta}$ are the stress and strain components. The elastic constants c_{ij} are considered to be thickness dependent, $c_{ij} = c_{ij}(\zeta)$, so that there is no need to consider the layer individually.

The kinematic relations in terms of the curvilinear coordinates can be expressed as

$$\begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\zeta \\ \gamma_{\beta\zeta} \\ \gamma_{\alpha\zeta} \\ \gamma_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} \partial_\alpha/\gamma_\alpha & 0 & 1/(\gamma_\alpha R_\alpha) \\ 0 & \partial_\beta/\gamma_\beta & 1/(\gamma_\beta R_\beta) \\ 0 & 0 & \partial_\zeta \\ 0 & \partial_\zeta - 1/(\gamma_\beta R_\beta) & \partial_\beta/\gamma_\beta \\ \partial_\zeta - 1/(\gamma_\alpha R_\alpha) & 0 & \partial_\alpha/\gamma_\alpha \\ \partial_\beta/\gamma_\beta & \partial_\alpha/\gamma_\alpha & 0 \end{bmatrix} \begin{Bmatrix} u_\alpha \\ u_\beta \\ u_\zeta \end{Bmatrix} \tag{2}$$

in which $\partial_\alpha = \partial/\partial\alpha$, $\partial_\beta = \partial/\partial\beta$, $\partial_\zeta = \partial/\partial\zeta$, $\gamma_\alpha = 1+\zeta/R_\alpha$, $\gamma_\beta = 1+\zeta/R_\beta$; u_α, u_β and u_ζ are the displacement components.

The equations of motion for the doubly curved shells with constant radii of curvature are

$$\gamma_\beta \frac{\partial\sigma_\alpha}{\partial\alpha} + \gamma_\alpha \frac{\partial\tau_{\alpha\beta}}{\partial\beta} + \gamma_\alpha\gamma_\beta \frac{\partial\tau_{\alpha\zeta}}{\partial\zeta} + \left(\frac{2}{R_\alpha} + \frac{1}{R_\beta} + \frac{3\zeta}{R_\alpha R_\beta}\right)\tau_{\alpha\zeta} = \gamma_\alpha\gamma_\beta\rho \frac{\partial^2 u_\alpha}{\partial t^2} \tag{3}$$

$$\gamma_\beta \frac{\partial\tau_{\alpha\beta}}{\partial\alpha} + \gamma_\alpha \frac{\partial\sigma_\beta}{\partial\beta} + \gamma_\alpha\gamma_\beta \frac{\partial\tau_{\beta\zeta}}{\partial\zeta} + \left(\frac{1}{R_\alpha} + \frac{2}{R_\beta} + \frac{3\zeta}{R_\alpha R_\beta}\right)\tau_{\beta\zeta} = \gamma_\alpha\gamma_\beta\rho \frac{\partial^2 u_\beta}{\partial t^2} \tag{4}$$

$$\gamma_\beta \frac{\partial\tau_{\alpha\zeta}}{\partial\alpha} + \gamma_\alpha \frac{\partial\tau_{\beta\zeta}}{\partial\beta} + \gamma_\alpha\gamma_\beta \frac{\partial\sigma_\zeta}{\partial\zeta} + \left(\frac{1}{R_\alpha} + \frac{1}{R_\beta} + \frac{2\zeta}{R_\alpha R_\beta}\right)\sigma_\zeta - \frac{\gamma_\beta}{R_\alpha}\sigma_\alpha - \frac{\gamma_\alpha}{R_\beta}\sigma_\beta = \gamma_\alpha\gamma_\beta\rho \frac{\partial^2 u_\zeta}{\partial t^2} \tag{5}$$

where ρ is the mass density that is a function of ζ, and t is the time variable.

The boundary conditions of the problem are specified as follows. On the lateral surfaces the transverse load $q(\alpha, \beta, t)$ is prescribed:

$$[\tau_{\alpha\zeta} \quad \tau_{\beta\zeta}] = [0 \quad 0] \quad \text{on } \zeta = \pm h \tag{6}$$

$$\sigma_\zeta = q(\alpha, \beta, t) \quad \text{on } \zeta = h \tag{7}$$

$$\sigma_\zeta = 0 \quad \text{on } \zeta = -h. \tag{8}$$

When the shell is open and of finite length, appropriate boundary conditions must be prescribed on the edges. It should be noted that the basic equations presented cannot be used for shells of general doubly curved surfaces such as the general conical, paraboloidal, toroidal shells, where the radii of curvature are not constant.

To facilitate the asymptotic analysis, we introduce the following dimensionless variables and multiple time scales in the formulation:

$$x = \frac{\alpha}{R\varepsilon}, \quad y = \frac{\beta}{R\varepsilon}, \quad z = \frac{\zeta}{h}, \quad u = \frac{u_\alpha}{R\varepsilon}, \quad v = \frac{u_\beta}{R\varepsilon}, \quad w = \frac{u_\zeta}{R}$$

$$R_x = \frac{R_\alpha}{R}, \quad R_y = \frac{R_\beta}{R}, \quad \sigma_x = \frac{\sigma_\alpha}{Q}, \quad \sigma_y = \frac{\sigma_\beta}{Q}, \quad \tau_{xy} = \frac{\tau_{\alpha\beta}}{Q}$$

$$\tau_{xz} = \frac{\tau_{\alpha\zeta}}{Q\varepsilon}, \quad \tau_{yz} = \frac{\tau_{\beta\zeta}}{Q\varepsilon}, \quad \sigma_z = \frac{\sigma_\zeta}{Q\varepsilon^2}, \quad \tau_k = \frac{\varepsilon^{2k}}{R}\sqrt{\frac{Q}{\rho_0}}\,t, \quad (k = 0, 1, 2, \ldots) \tag{9}$$

where $\varepsilon^2 = h/R < 1$, R denotes a characteristic length of the shell, Q and ρ_0 represent a reference elastic modulus and a reference mass density, respectively.

As presented earlier (Wu *et al.*, 1996a, b), we shall take the displacements u_α, u_β, u_ζ and transverse stresses $\tau_{\alpha\zeta}$, $\tau_{\beta\zeta}$, σ_ζ as the primary variables, the membrane stresses σ_α, σ_β and $\tau_{\alpha\beta}$ as the dependent variables. Upon reformulation and nondimensionalization, we can express the basic equations in dimensionless forms as follows:

$$w_{,z} = -\varepsilon^2 \mathbf{L}_1 \mathbf{u} - \varepsilon^2 \tilde{l}_{33} w + \varepsilon^4 \tilde{l}_{34} \sigma_z \tag{10}$$

$$\mathbf{u}_{,z} = -\mathbf{D}w + \varepsilon^2 \mathbf{L}_2 \mathbf{u} + \varepsilon^2 \mathbf{S}\boldsymbol{\sigma}_s + \varepsilon^4 \mathbf{L}_3 \boldsymbol{\sigma}_s \tag{11}$$

$$\boldsymbol{\sigma}_m = \mathbf{L}_4 \mathbf{u} + \mathbf{L}_5 w + \varepsilon^2 \mathbf{L}_6 \sigma_z \tag{12}$$

$$\boldsymbol{\sigma}_{s,z} = -\mathbf{L}_7\mathbf{u}-\mathbf{L}_8 w-\varepsilon^2\mathbf{L}_9\boldsymbol{\sigma}_s-\varepsilon^2\mathbf{L}_{10}\sigma_z-\varepsilon^4\mathbf{L}_{11}\boldsymbol{\sigma}_s$$
$$+\rho_1\left[\frac{\partial^2}{\partial\tau_0^2}+2\varepsilon^2\frac{\partial^2}{\partial\tau_0\,\partial\tau_1}+\varepsilon^4\left(2\frac{\partial^2}{\partial\tau_0\,\partial\tau_2}+\frac{\partial^2}{\partial\tau_1^2}\right)+\cdots\right]\mathbf{u} \quad (13)$$

$$\sigma_{z,z} = \mathbf{L}_8^{\mathrm{T}}\mathbf{u}+\tilde{l}_{63}w-\mathbf{D}^{\mathrm{T}}\boldsymbol{\sigma}_s-\varepsilon^2\mathbf{L}_{12}\boldsymbol{\sigma}_s-\varepsilon^2\tilde{l}_{64}\sigma_z-\varepsilon^4\tilde{l}_{65}\sigma_z$$
$$+\rho_2\left[\frac{\partial^2}{\partial\tau_0^2}+2\varepsilon^2\frac{\partial^2}{\partial\tau_0\,\partial\tau_1}+\varepsilon^4\left(2\frac{\partial^2}{\partial\tau_0\,\partial\tau_2}+\frac{\partial^2}{\partial\tau_1^2}\right)+\cdots\right]w \quad (14)$$

where

$$\mathbf{u} = \{u \quad v]^{\mathrm{T}},\quad \boldsymbol{\sigma}_s = \{\tau_{xz} \quad \tau_{yz}\}^{\mathrm{T}},\quad \boldsymbol{\sigma}_m = \{\sigma_x \quad \sigma_y \quad \tau_{xy}\}^{\mathrm{T}},\quad \mathbf{D} = \{\partial_x \quad \partial_y\}^{\mathrm{T}}$$
$$\rho_1 = \frac{h\rho}{R\rho_0},\quad \rho_2 = \left(1+\frac{hz}{RR_x}\right)\left(1+\frac{hz}{RR_y}\right)\frac{\rho}{\rho_0},\quad \mathbf{L}_1 = [\tilde{l}_{31} \quad \tilde{l}_{32}]$$

$$\mathbf{L}_2 = \begin{bmatrix}\tilde{l}_{11} & 0\\ 0 & \tilde{l}_{22}\end{bmatrix},\quad \mathbf{S} = \begin{bmatrix}\tilde{l}_{14} & \tilde{l}_{15}\\ \tilde{l}_{15} & \tilde{l}_{25}\end{bmatrix},\quad \mathbf{L}_3 = \begin{bmatrix}\tilde{l}_{16} & \tilde{l}_{17}\\ \tilde{l}_{26} & \tilde{l}_{27}\end{bmatrix},\quad \mathbf{L}_4 = \begin{bmatrix}\tilde{l}_{71} & \tilde{l}_{72}\\ \tilde{l}_{81} & \tilde{l}_{82}\\ \tilde{l}_{91} & \tilde{l}_{92}\end{bmatrix}$$

$$\mathbf{L}_5 = \begin{bmatrix}\tilde{l}_{73}\\ \tilde{l}_{83}\\ \tilde{l}_{93}\end{bmatrix},\quad \mathbf{L}_6 = \begin{bmatrix}\tilde{c}_{13}\\ \tilde{c}_{23}\\ \tilde{c}_{36}\end{bmatrix},\quad \mathbf{L}_7 = \begin{bmatrix}\tilde{l}_{41} & \tilde{l}_{42}\\ \tilde{l}_{42} & \tilde{l}_{52}\end{bmatrix},\quad \mathbf{L}_8 = \begin{bmatrix}\tilde{l}_{43}\\ \tilde{l}_{53}\end{bmatrix}$$

$$\mathbf{L}_9 = \begin{bmatrix}\tilde{l}_{44} & 0\\ 0 & \tilde{l}_{55}\end{bmatrix},\quad \mathbf{L}_{10} = \begin{bmatrix}\tilde{l}_{46}\\ \tilde{l}_{56}\end{bmatrix},\quad \mathbf{L}_{11} = \begin{bmatrix}\tilde{l}_{45} & 0\\ 0 & \tilde{l}_{45}\end{bmatrix},\quad \mathbf{L}_{12} = [\tilde{l}_{61} \quad \tilde{l}_{62}].$$

The expressions of $\tilde{l}_{ij}$ are the same as those defined in the previous paper (Wu *et al.*, 1996b).

2.2. *Asymptotic expansion*

Since eqns (10)–(14) involve terms associated with even powers of ε only, we expand the displacements and stresses in powers ε^2 as given by

$$f = f_{(0)}(x,y,z,\tau_0,\tau_1,\ldots)+\varepsilon^2 f_{(1)}(x,y,z,\tau_0,\tau_1,\ldots)+\varepsilon^4 f_{(2)}(x,y,z,\tau_0,\tau_1,\ldots)+,\ldots. \quad (15)$$

After substituting eqn (15) into eqns (10)–(14) and collecting terms of equal powers of ε, the basic equations are decomposed into simpler sets of differential equations that can be integrated with respect to z in succession. It has been shown that the asymptotic integration leads to the CST equations at each level of approximation.

To derive an asymptotic theory leading to the FSDT and HSDT equations at each level, let us examine the basic equations in connection with the perturbation parameter. In eqn (10) the transverse displacement is not explicitly related to the transverse shear stresses, whereas in eqn (11) the in-surface displacements are related to them at the ε^2 level. It is at the ε^2-order that the transverse shear deformations are called upon to the stage. In order to account for the effect of transverse shear deformations at the leading-order, we introduce the auxiliary variables ψ_x and ψ_y associated with the transverse shear deformations and express the transverse shear strains as

$$\varepsilon^2\mathbf{S}\boldsymbol{\sigma}_s = (1-\lambda z^2)\boldsymbol{\psi}+\varepsilon^2\mathbf{S}\hat{\boldsymbol{\sigma}}_s \quad (16)$$

where $\boldsymbol{\psi} = \{\psi_x,\psi_y\}^{\mathrm{T}}$, $\hat{\boldsymbol{\sigma}}_s = \{\hat{\tau}_{xz},\hat{\tau}_{yz}\}^{\mathrm{T}}$, ψ_x and ψ_y represent the average shear rotations at the middle surface, which are independent of z by definition, $\hat{\tau}_{xz}$ and $\hat{\tau}_{yz}$ are the difference between the actual transverse shear stresses and the approximate ones. In extracting the shear rotations from the transverse shear deformations, as given in eqn (16), we are able to consider the effect of the transverse shears right from the leading-order level. The remaining contribution of the transverse shear deformations, as denoted by $\mathbf{S}\hat{\boldsymbol{\sigma}}_s$, are considered to be higher-orders and will be treated subsequently. It will be shown that the decomposition in eqn (16) with $\lambda = 0$ and 1 lead to the FSDT and HSDT equations, respectively.

With the shear rotations as auxiliary variables, the moment equilibrium equations must be considered. Two more equations are obtained by multiplying eqn (13) by $(z-\lambda z^3/3)$, integrating over the thickness, using the integration by parts and lateral boundary conditions eqns (6)–(8). It follows that

$$\int_{-1}^{1} \boldsymbol{\sigma}_s(1-\lambda z^2)\,dz = \int_{-1}^{1}\left(z-\lambda\frac{z^3}{3}\right)(\mathbf{L}_7\mathbf{u}+\mathbf{L}_8 w)\,dz+\varepsilon^2\int_{-1}^{1}\left(z-\lambda\frac{z^3}{3}\right)(\mathbf{L}_9\boldsymbol{\sigma}_s+\mathbf{L}_{10}\sigma_z)\,dz$$

$$+\varepsilon^4\int_{-1}^{1}\left(z-\lambda\frac{z^3}{3}\right)\mathbf{L}_{11}\boldsymbol{\sigma}_s\,dz-\int_{-1}^{1}\left[\rho_1\left(z-\lambda\frac{z^3}{3}\right)\left(\frac{\partial^2}{\partial\tau_0^2}+2\varepsilon^2\frac{\partial^2}{\partial\tau_0\,\partial\tau_1}+\cdots\right)\mathbf{u}\right]dz. \quad (17)$$

Substituting eqn (15) into eqns (10)–(14), (17) and collecting terms of equal powers of ε, we obtain the following sets of equations.

Order ε^0:

$$w_{(0),z} = 0 \quad (18)$$

$$\mathbf{u}_{(0),z} = -\mathbf{D}w_{(0)}+(1-\lambda z^2)\boldsymbol{\psi}_0 \quad (19)$$

$$\boldsymbol{\sigma}_{m(0)} = \mathbf{L}_4\mathbf{u}_{(0)}+\mathbf{L}_5 w_{(0)} \quad (20)$$

$$\boldsymbol{\sigma}_{s(0),z} = -\mathbf{L}_7\mathbf{u}_{(0)}-\mathbf{L}_8 w_{(0)}+\rho_1\frac{\partial^2}{\partial\tau_0^2}\mathbf{u}_{(0)} \quad (21)$$

$$\sigma_{z(0),z} = \mathbf{L}_8^T\mathbf{u}_{(0)}+\tilde{l}_{63}w_{(0)}-\mathbf{D}^T\boldsymbol{\sigma}_{s(0)}+\rho_2\frac{\partial^2}{\partial\tau_0^2}w_{(0)} \quad (22)$$

$$\int_{-1}^{1}(1-\lambda z^2)^2\mathbf{C}_s\boldsymbol{\psi}_0\,dz = \int_{-1}^{1}\left(z-\lambda\frac{z^3}{3}\right)(\mathbf{L}_7\mathbf{u}_{(0)}+\mathbf{L}_8 w_{(0)})\,dz-\int_{-1}^{1}\rho_1\left(z-\lambda\frac{z^3}{3}\right)\frac{\partial^2\mathbf{u}_{(0)}}{\partial\tau_0^2}dz. \quad (23)$$

Order ε^2:

$$w_{(1),z} = -\mathbf{L}_1\mathbf{u}_{(0)}-\tilde{l}_{33}w_{(0)} \quad (24)$$

$$\mathbf{u}_{(1),z} = \mathbf{L}_2\mathbf{u}_{(0)}-\mathbf{D}w_{(1)}+\mathbf{S}\hat{\boldsymbol{\sigma}}_{s(0)}+(1-\lambda z^2)\boldsymbol{\psi}_1 \quad (25)$$

$$\boldsymbol{\sigma}_{m(1)} = \mathbf{L}_4\mathbf{u}_{(1)}+\mathbf{L}_5 w_{(1)}+\mathbf{L}_6\sigma_{z(0)} \quad (26)$$

$$\boldsymbol{\sigma}_{s(1),z} = -\mathbf{L}_7\mathbf{u}_{(1)}-\mathbf{L}_8 w_{(1)}-\mathbf{L}_9\boldsymbol{\sigma}_{s(0)}-\mathbf{L}_{10}\sigma_{z(0)}+\rho_1\frac{\partial^2}{\partial\tau_0^2}\mathbf{u}_{(1)}+2\rho_1\frac{\partial^2}{\partial\tau_0\,\partial\tau_1}\mathbf{u}_{(0)} \quad (27)$$

$$\sigma_{z(1),z} = \mathbf{L}_8^T\mathbf{u}_{(1)}+\tilde{l}_{63}w_{(1)}-\mathbf{D}^T\boldsymbol{\sigma}_{s(1)}-\mathbf{L}_{12}\boldsymbol{\sigma}_{s(0)}-\tilde{l}_{64}\sigma_{z(0)}+\rho_2\frac{\partial^2}{\partial\tau_0^2}w_{(1)}+2\rho_2\frac{\partial^2}{\partial\tau_0\,\partial\tau_1}w_{(0)} \quad (28)$$

$$\int_{-1}^{1}(1-\lambda z^2)^2\mathbf{C}_s\boldsymbol{\psi}_1\,\mathrm{d}z=\int_{-1}^{1}\left(z-\lambda\frac{z^3}{3}\right)(\mathbf{L}_7\mathbf{u}_{(1)}+\mathbf{L}_8 w_{(1)})\,\mathrm{d}z$$

$$+\int_{-1}^{1}\left(z-\lambda\frac{z^3}{3}\right)(\mathbf{L}_9\boldsymbol{\sigma}_{s(0)}+\mathbf{L}_{10}\sigma_{z(0)})\,\mathrm{d}z-\int_{-1}^{1}\rho_1\left(z-\lambda\frac{z^3}{3}\right)\left(\frac{\partial^2\mathbf{u}_{(1)}}{\partial\tau_0^2}+2\frac{\partial^2\mathbf{u}_{(0)}}{\partial\tau_0\,\partial\tau_1}\right)\mathrm{d}z \quad (29)$$

where $\mathbf{C}_s=(R/h)\mathbf{S}^{-1}$.

The dimensionless lateral boundary conditions associated with each level are

Order ε^0:

$$[\tau_{xz(0)}\quad\tau_{yz(0)}]=[0\quad 0]\quad\text{on } z=\pm 1 \quad (30)$$

$$\sigma_{z(0)}=\tilde{q}(x,y,t)\quad\text{on } z=1 \quad (31)$$

$$\sigma_{z(0)}=0\quad\text{on } z=-1 \quad (32)$$

Order ε^{2k}: $(k=1,2,3,\ldots)$

$$[\tau_{xz(k)}\quad\tau_{yz(k)}]=[0\quad 0]\quad\text{on } z=\pm 1 \quad (33)$$

$$\sigma_{z(k)}=0\quad\text{on } z=\pm 1 \quad (34)$$

where $\tilde{q}=q/Q\varepsilon^2$.

The higher-order equations can be written out, but are not given here for brevity.

3. SUCCESSIVE INTEGRATION

The asymptotic equations can be integrated with respect to z in succession. The associated lateral boundary conditions will be satisfied along the way. As a result, we obtain

$$w_{(0)}=w_0(x,y,\tau_0,\tau_1,\ldots), \quad (35)$$

$$\mathbf{u}_{(0)}=\mathbf{u}_0(x,y,\tau_0,\tau_1,\ldots)+z\boldsymbol{\phi}_0-\frac{\lambda z^3}{3}(\boldsymbol{\phi}_0+\mathbf{D}w_0), \quad (36)$$

$$\boldsymbol{\sigma}_{s(0)}=-\int_{-1}^{z}\left\{\mathbf{L}_7\left[\mathbf{u}_0+\eta\boldsymbol{\phi}_0-\frac{\lambda\eta^3}{3}(\boldsymbol{\phi}_0+\mathbf{D}w_0)\right]+\mathbf{L}_8 w_0\right\}\mathrm{d}\eta$$

$$+\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z}\rho_1\left[\mathbf{u}_0+\eta\boldsymbol{\phi}_0-\frac{\lambda\eta^3}{3}(\boldsymbol{\phi}_0+\mathbf{D}w_0)\right]\mathrm{d}\eta \quad (37)$$

$$\sigma_{z(0)}=\int_{-1}^{z}\left\{\mathbf{L}_8^{\mathrm{T}}\left[\mathbf{u}_0+\eta\boldsymbol{\phi}_0-\frac{\lambda\eta^3}{3}(\boldsymbol{\phi}_0+\mathbf{D}w_0)\right]+\tilde{l}_{63}w_0\right\}\mathrm{d}\eta$$

$$+\int_{-1}^{z}(z-\eta)\mathbf{D}^{\mathrm{T}}\left\{\mathbf{L}_7\left[\mathbf{u}_0+\eta\boldsymbol{\phi}_0-\frac{\lambda\eta^3}{3}(\boldsymbol{\phi}_0+\mathbf{D}w_0)\right]+\mathbf{L}_8 w_0\right\}\mathrm{d}\eta$$

$$+\int_{-1}^{z}\rho_2\frac{\partial^2 w_0}{\partial\tau_0^2}\mathrm{d}\eta-\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z}\rho_1(z-\eta)\mathbf{D}^{\mathrm{T}}\left[\mathbf{u}_0+\eta\boldsymbol{\phi}_0-\frac{\lambda\eta^3}{3}(\boldsymbol{\phi}_0+\mathbf{D}w_0)\right]\mathrm{d}\eta \quad (38)$$

where $\boldsymbol{\phi}_0=\{\phi_{x(0)},\phi_{y(0)}\}^{\mathrm{T}}=\boldsymbol{\psi}_0-\mathbf{D}w_0$, w_0 and $\mathbf{u}_0=\{u_0,v_0\}^{\mathrm{T}}$ are the displacements on the middle surface.

The lateral boundary conditions are considered next. The boundary conditions on $z = -1$ are identically satisfied by eqns (37) and (38). Imposition of the remaining conditions on eqns (37)–(38) and substitution of eqns (35) and (36) in eqn (23) yields

$$\int_{-1}^{1}\left\{\mathbf{L}_7\left[\mathbf{u}_0+z\boldsymbol{\phi}_0-\frac{\lambda z^3}{3}(\boldsymbol{\phi}_0+\mathbf{D}w_0)\right]+\mathbf{L}_8 w_0\right\}\mathrm{d}z$$
$$-\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{1}\rho_1\left[\mathbf{u}_0+z\boldsymbol{\phi}_0-\frac{\lambda z^3}{3}(\boldsymbol{\phi}_0+\mathbf{D}w_0)\right]\mathrm{d}z = 0 \quad (39)$$

$$\int_{-1}^{1}\left\{\mathbf{L}_8^{\mathrm{T}}\left[\mathbf{u}_0+z\boldsymbol{\phi}_0-\frac{\lambda z^3}{3}(\boldsymbol{\phi}_0+\mathbf{D}w_0)\right]+\tilde{l}_{63}w_0\right\}\mathrm{d}z$$
$$+\int_{-1}^{1}(1-z)\mathbf{D}^{\mathrm{T}}\left\{\mathbf{L}_7\left[\mathbf{u}_0+z\boldsymbol{\phi}_0-\frac{\lambda z^3}{3}(\boldsymbol{\phi}_0+\mathbf{D}w_0)\right]+\mathbf{L}_8 w_0\right\}\mathrm{d}z$$
$$+\int_{-1}^{1}\rho_2\frac{\partial^2 w_0}{\partial\tau_0^2}\mathrm{d}\eta-\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{1}\rho_1(1-z)\mathbf{D}^{\mathrm{T}}\left[\mathbf{u}_0+z\boldsymbol{\phi}_0-\frac{\lambda z^3}{3}(\boldsymbol{\phi}_0+\mathbf{D}w_0)\right]\mathrm{d}z = \tilde{q} \quad (40)$$

$$\int_{-1}^{1}(1-\lambda z^2)^2\mathbf{C}_s(\boldsymbol{\phi}_0+\mathbf{D}w_0)\,\mathrm{d}z = \int_{-1}^{1}\left(z-\lambda\frac{z^3}{3}\right)[\mathbf{L}_7\mathbf{u}_{(0)}+\mathbf{L}_8 w_{(0)}]\,\mathrm{d}z$$
$$-\int_{-1}^{1}\rho_1\left(z-\lambda\frac{z^3}{3}\right)\frac{\partial^2\mathbf{u}_{(0)}}{\partial\tau_0^2}\mathrm{d}z. \quad (41)$$

After simplifying eqn (40) using eqns (40) and (41), these equations can be expressed as

$$\mathbf{K}\boldsymbol{\Delta}_0 = \mathbf{M}\frac{\partial^2\boldsymbol{\Delta}_0}{\partial\tau_0^2}+\mathbf{F}_0 \quad (42)$$

where $\boldsymbol{\Delta}_0 = \{u_0, v_0, w_0, \phi_{x(0)}, \phi_{y(0)}\}^{\mathrm{T}}$, $\mathbf{F}_0 = \{0,0,\tilde{q},0,0\}^{\mathrm{T}}$

$$K_{11} = \hat{A}_{11}\,\partial_{xx}+2\tilde{A}_{16}\,\partial_{xy}+\bar{A}_{66}\,\partial_{yy},\quad K_{12} = K_{21} = \hat{A}_{16}\,\partial_{xx}+(\tilde{A}_{12}+\tilde{A}_{66})\,\partial_{xy}+\bar{A}_{26}\,\partial_{yy}$$

$$K_{13} = K_{31} = \left(\frac{\hat{A}_{11}}{R_x}+\frac{\tilde{A}_{12}}{R_y}\right)\partial_x+\left(\frac{\tilde{A}_{16}}{R_x}+\frac{\bar{A}_{26}}{R_y}\right)\partial_y$$
$$-\lambda\left[\frac{\hat{E}_{11}}{3}\partial_{xxx}+\frac{(2\tilde{E}_{16}+\hat{E}_{16})}{3}\partial_{xxy}+\frac{(\bar{E}_{66}+\tilde{E}_{12}+\tilde{E}_{66})}{3}\partial_{xyy}+\frac{\bar{E}_{26}}{3}\partial_{yyy}\right]$$

$$K_{14} = K_{41} = \hat{B}_{11}\,\partial_{xx}+2\tilde{B}_{16}\,\partial_{xy}+\bar{B}_{66}\,\partial_{yy}-\lambda\left[\frac{\hat{E}_{11}}{3}\partial_{xx}+\frac{2\tilde{E}_{16}}{3}\partial_{xy}+\frac{\bar{E}_{66}}{3}\partial_{yy}\right]$$

$$K_{15} = K_{51} = \hat{B}_{16}\,\partial_{xx}+(\tilde{B}_{12}+\tilde{B}_{66})\,\partial_{xy}+\bar{B}_{26}\,\partial_{yy}-\lambda\left[\frac{\hat{E}_{16}}{3}\partial_{xx}+\frac{(\tilde{E}_{12}+\tilde{E}_{66})}{3}\partial_{xy}+\frac{\bar{E}_{26}}{3}\partial_{yy}\right]$$

$$K_{22} = \hat{A}_{66}\,\partial_{xx}+2\tilde{A}_{26}\,\partial_{xy}+\bar{A}_{22}\,\partial_{yy}$$

$$K_{23} = K_{32} = \left(\frac{\hat{A}_{16}}{R_x}+\frac{\tilde{A}_{26}}{R_y}\right)\partial_x+\left(\frac{\tilde{A}_{12}}{R_x}+\frac{\bar{A}_{22}}{R_y}\right)\partial_y$$
$$-\lambda\left[\frac{\hat{E}_{16}}{3}\partial_{xxx}+\frac{(\tilde{E}_{12}+\tilde{E}_{66}+\hat{E}_{66})}{3}\partial_{xxy}+\frac{(\bar{E}_{26}+2\tilde{E}_{26})}{3}\partial_{xyy}+\frac{\bar{E}_{22}}{3}\partial_{yyy}\right]$$

$$K_{24} = K_{42} = K_{15}$$

$$K_{25} = \hat{B}_{66}\,\partial_{xx} + 2\tilde{B}_{26}\,\partial_{xy} + \bar{B}_{22}\,\partial_{yy} - \lambda\left[\frac{\hat{E}_{66}}{3}\,\partial_{xx} + \frac{2\tilde{E}_{26}}{3}\,\partial_{xy} + \frac{\bar{E}_{22}}{3}\,\partial_{yy}\right]$$

$$\begin{aligned}
K_{33} = {} & \frac{\hat{A}_{11}}{R_x^2} + \frac{2\tilde{A}_{12}}{R_xR_y} + \frac{\bar{A}_{22}}{R_y^2} - \tilde{A}_{55}\,\partial_{xx} - 2\tilde{A}_{45}\,\partial_{xy} - \tilde{A}_{44}\,\partial_{yy} \\
& - \lambda\left[\frac{2}{3}\left(\frac{\hat{E}_{11}}{R_x} + \frac{\tilde{E}_{12}}{R_y}\right)\partial_{xx} + \frac{2}{3}\left(\frac{\tilde{E}_{16}}{R_x} + \frac{\bar{E}_{26}}{R_y} + \frac{\hat{E}_{16}}{R_x} + \frac{\tilde{E}_{26}}{R_y}\right)\partial_{xy}\right. \\
& + \frac{2}{3}\left(\frac{\tilde{E}_{12}}{R_x} + \frac{\bar{E}_{22}}{R_y}\right)\partial_{yy} - \frac{\hat{H}_{11}}{9}\,\partial_{xxxx} - \frac{2(\tilde{H}_{16} + \hat{H}_{16})}{9}\,\partial_{xxxy} \\
& - \frac{(2\tilde{H}_{12} + \bar{H}_{66} + \hat{H}_{66} + 2\tilde{H}_{66})}{9}\,\partial_{xxyy} - \frac{2(\bar{H}_{26} + \tilde{H}_{26})}{9}\,\partial_{xyyy} \\
& \left. - \frac{\bar{H}_{22}}{9}\,\partial_{yyyy} - (2\tilde{D}_{55} - \tilde{F}_{55})\,\partial_{xx} - 2(2\tilde{D}_{45} - \tilde{F}_{45})\,\partial_{xy} - (2\tilde{D}_{44} - \tilde{F}_{44})\,\partial_{yy}\right],
\end{aligned}$$

$$\begin{aligned}
K_{34} = {} & \left(\frac{\hat{B}_{11}}{R_x} + \frac{\tilde{B}_{12}}{R_y}\right)\partial_x + \left(\frac{\tilde{B}_{16}}{R_x} + \frac{\bar{B}_{26}}{R_y}\right)\partial_y - \tilde{A}_{55}\,\partial_x - \tilde{A}_{45}\,\partial_y \\
& - \lambda\left[\left(\frac{\hat{E}_{11}}{3R_x} + \frac{\tilde{E}_{12}}{3R_y}\right)\partial_x + \left(\frac{\tilde{E}_{16}}{3R_x} + \frac{\bar{E}_{26}}{3R_y}\right)\partial_y\right. \\
& + \frac{1}{3}\left(\hat{F}_{11} - \frac{\hat{H}_{11}}{3}\right)\partial_{xxx} + \frac{1}{3}\left(2\tilde{F}_{16} - \frac{2}{3}\tilde{H}_{16} + \hat{F}_{16} - \frac{\hat{H}_{16}}{3}\right)\partial_{xxy} \\
& + \frac{1}{3}\left(\bar{F}_{66} - \frac{\bar{H}_{66}}{3} + \tilde{F}_{12} - \frac{\tilde{H}_{12}}{3} + \tilde{F}_{66} - \frac{\tilde{H}_{66}}{3}\right)\partial_{xyy} \\
& \left. + \frac{1}{3}\left(\bar{F}_{26} - \frac{\bar{H}_{26}}{3}\right)\partial_{yyy} - (2\tilde{D}_{55} - \tilde{F}_{55})\,\partial_x - (2\tilde{D}_{45} - \tilde{F}_{45})\,\partial_y\right],
\end{aligned}$$

$$\begin{aligned}
K_{35} = {} & \left(\frac{\hat{B}_{16}}{R_x} + \frac{\tilde{B}_{26}}{R_y}\right)\partial_x + \left(\frac{\tilde{B}_{12}}{R_x} + \frac{\bar{B}_{22}}{R_y}\right)\partial_y - \tilde{A}_{45}\,\partial_x - \tilde{A}_{44}\,\partial_y \\
& - \lambda\left[\left(\frac{\hat{E}_{16}}{3R_x} + \frac{\tilde{E}_{26}}{3R_y}\right)\partial_x + \left(\frac{\tilde{E}_{12}}{3R_x} + \frac{\bar{E}_{22}}{3R_y}\right)\partial_y\right. \\
& + \frac{1}{3}\left(\hat{F}_{16} - \frac{\hat{H}_{16}}{3}\right)\partial_{xxx} + \frac{1}{3}\left(\tilde{F}_{12} - \frac{\tilde{H}_{12}}{3} + \tilde{F}_{66} - \frac{\tilde{H}_{66}}{3} + \hat{F}_{66} - \frac{\hat{H}_{66}}{3}\right)\partial_{xxy} \\
& + \frac{1}{3}\left(2\tilde{F}_{26} - \frac{2}{3}\tilde{H}_{26} + \bar{F}_{26} - \frac{\bar{H}_{26}}{3}\right)\partial_{xyy} \\
& \left. + \frac{1}{3}\left(\bar{F}_{22} - \frac{\bar{H}_{22}}{3}\right)\partial_{yyy} - (2\tilde{D}_{45} - \tilde{F}_{45})\,\partial_x - (2\tilde{D}_{44} - \tilde{F}_{44})\,\partial_y\right],
\end{aligned}$$

$$\begin{aligned}
K_{44} = {} & \hat{D}_{11}\,\partial_{xx} + 2\tilde{D}_{16}\,\partial_{xy} + \bar{D}_{66}\,\partial_{yy} - \tilde{A}_{55} - \lambda[(\tfrac{2}{3}\hat{F}_{11} - \tfrac{1}{9}\hat{H}_{11})\,\partial_{xx} + 2(\tfrac{2}{3}\tilde{F}_{16} - \tfrac{1}{9}\tilde{H}_{16})\,\partial_{xy} \\
& + (\tfrac{2}{3}\bar{F}_{66} - \tfrac{1}{9}\bar{H}_{66})\,\partial_{yy} - (2\tilde{D}_{55} - \tilde{F}_{55})],
\end{aligned}$$

$$\begin{aligned}
K_{45} = K_{54} = {} & \hat{D}_{16}\,\partial_{xx} + (\tilde{D}_{12} + \tilde{D}_{66})\,\partial_{xy} + \bar{D}_{26}\,\partial_{yy} - \tilde{A}_{45} - \lambda[(\tfrac{2}{3}\hat{F}_{16} - \tfrac{1}{9}\hat{H}_{16})\,\partial_{xx} \\
& + (\tfrac{2}{3}\tilde{F}_{12} - \tfrac{1}{9}\tilde{H}_{12} + \tfrac{2}{3}\tilde{F}_{66} - \tfrac{1}{9}\tilde{H}_{66})\,\partial_{xy} + (\tfrac{2}{3}\bar{F}_{26} - \tfrac{1}{9}\bar{H}_{26})\,\partial_{yy} + 2\tilde{D}_{45} - \tilde{F}_{45}],
\end{aligned}$$

$$K_{55} = \hat{D}_{66}\,\partial_{xx} + 2\tilde{D}_{26}\,\partial_{xy} + \bar{D}_{22}\,\partial_{yy} - \tilde{A}_{44} - \lambda[(\tfrac{2}{3}\hat{F}_{66} - \tfrac{1}{9}\hat{H}_{66})\,\partial_{xx} + 2(\tfrac{2}{3}\tilde{F}_{26} - \tfrac{1}{9}\tilde{H}_{26})\,\partial_{xy}$$

$$+(\tfrac{2}{3}\tilde{F}_{22}-\tfrac{1}{9}\tilde{H}_{22})\,\partial_{yy}-(2\tilde{D}_{44}-\tilde{F}_{44})],$$

$$M_{11}=M_{22}=I_{10},\quad M_{13}=M_{31}=-\lambda\frac{I_{13}}{3}\partial_x,\quad M_{14}=M_{41}=M_{25}=M_{52}=I_{11}-\lambda\frac{I_{13}}{3},$$

$$M_{23}=M_{32}=-\lambda\frac{I_{13}}{3}\partial_y,\quad M_{33}=-I_{20}+\lambda\frac{I_{16}}{9}(\partial_{xx}+\partial_{yy}),$$

$$M_{34}=M_{43}=-\lambda\left(\frac{I_{14}}{3}-\frac{I_{16}}{9}\right)\partial_x,$$

$$M_{35}=M_{53}=-\lambda\left(\frac{I_{14}}{3}-\frac{I_{16}}{9}\right)\partial_y,\quad M_{44}=M_{55}=I_{12}-\lambda\left(\frac{2I_{14}}{3}-\frac{I_{16}}{9}\right),$$

$$M_{12}=M_{21}=M_{15}=M_{51}=M_{24}=M_{42}=M_{45}=M_{54}=0,$$

$$(\hat{A}_{ij},\hat{B}_{ij},\hat{D}_{ij},\hat{E}_{ij},\hat{F}_{ij},\hat{H}_{ij})=\int_{-1}^{1}\frac{\tilde{Q}_{ij}\gamma_\beta}{\gamma_\alpha}(1,z,z^2,z^3,z^4,z^6)\,dz\quad(i,j=1,2,6),$$

$$(\tilde{A}_{ij},\tilde{B}_{ij},\tilde{D}_{ij},\tilde{E}_{ij},\tilde{F}_{ij},\tilde{H}_{ij})=\int_{-1}^{1}\tilde{Q}_{ij}(1,z,z^2,z^3,z^4,z^6)\,dz\quad(i,j=1,2,6),$$

$$(\bar{A}_{ij},\bar{B}_{ij},\bar{D}_{ij},\bar{E}_{ij},\bar{F}_{ij},\bar{H}_{ij})=\int_{-1}^{1}\frac{\tilde{Q}_{ij}\gamma_\alpha}{\gamma_\beta}(1,z,z^2,z^3,z^4,z^6)\,dz\quad(i,j=1,2,6),$$

$$(\tilde{A}_{ij},\tilde{B}_{ij},\tilde{D}_{ij},\tilde{E}_{ij},\tilde{F}_{ij},\tilde{H}_{ij})=\int_{-1}^{1}\tilde{c}_{ij}(1,z,z^2,z^3,z^4,z^6)\,dz\quad(i,j=4,5),$$

$$(I_{10},I_{11},I_{12},I_{13},I_{14},I_{16})=\int_{-1}^{1}\rho_1(1,z,z^2,z^3,z^4,z^6)\,dz,\quad I_{20}=\int_{-1}^{1}\rho_2\,dz.$$

The governing equations for the displacements in FSDT (Khdeir *et al.*, 1989) and HSDT (Reddy and Liu, 1985) are reproduced by introducing a geometry assumption of the thin shell in eqn (42): $z/R_\alpha \ll 1$ and $z/R_\beta \ll 1$. In the present notation, this implies $\gamma_\alpha \approx 1$, $\gamma_\beta \approx 1$, $\bar{A}_{ij}=\hat{A}_{ij}=\tilde{A}_{ij}=A_{ij}/Qh$, $\bar{B}_{ij}=\hat{B}_{ij}=\tilde{B}_{ij}=B_{ij}/Qh^2$, $\bar{D}_{ij}=\hat{D}_{ij}=\tilde{D}_{ij}=D_{ij}/Qh^3$, $\bar{E}_{ij}=\hat{E}_{ij}=\tilde{E}_{ij}=E_{ij}/Qh^4$, $\bar{F}_{ij}=\hat{F}_{ij}=\tilde{F}_{ij}=F_{ij}/Qh^5$ and $\bar{H}_{ij}=\hat{H}_{ij}=\tilde{H}_{ij}=H_{ij}/Qh^7$. The coefficients A_{ij}, B_{ij}, D_{ij} are the extension, extension-bending and bending stiffness, respectively. E_{ij}, F_{ij}, H_{ij} are the higher-order influence coefficients in HSDT. Thus, the FSDT and HSDT equations have been derived as the leading-order approximation to the three-dimensional (3-D) theory. Obviously, these two-dimensional (2-D) approximate shell theories are usually inadequate for thick shells since they amount to the leading-order approximation. Note that there is no need to use the shear correction factor in the present formulation.

The solution of eqn (42) must be supplemented with the appropriate edge boundary conditions. Once u_0, v_0, w_0, ϕ_{x0} and ϕ_{y0} are determined, the leading-order displacements can be obtained from eqns (35) and (36), the transverse stresses from eqns (37) and (38), and the membrane stresses from eqn (20).

Proceeding to order ε^2, we readily obtain

$$w_{(1)}=w_1(x,y,\tau_0,\tau_1,\ldots)+\varphi_{31}(x,y,z,\tau_0,\tau_1,\ldots)\tag{43}$$

$$\mathbf{u}_{(1)}=\mathbf{u}_1(x,y,\tau_0,\tau_1,\ldots)+z\boldsymbol{\phi}_1-\frac{\lambda z^3}{3}(\boldsymbol{\phi}_1+\mathbf{D}w_1)+\boldsymbol{\varphi}_1(x,y,z,\tau_0,\tau_1,\ldots)\tag{44}$$

$$\boldsymbol{\sigma}_{s(1)}=-\int_{-1}^{z}\left\{\mathbf{L}_7\left[\mathbf{u}_1+\eta\boldsymbol{\phi}_1-\frac{\lambda\eta^3}{3}(\boldsymbol{\phi}_1+\mathbf{D}w_1)\right]+\mathbf{L}_8w_1\right\}d\eta+\mathbf{f}_1(x,y,z,\tau_0,\tau_1,\ldots)$$

$$+\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z}\rho_1\left[\mathbf{u}_1+\eta\boldsymbol{\phi}_1-\frac{\lambda\eta^3}{3}(\boldsymbol{\phi}_1+\mathbf{D}w_1)\right]\mathrm{d}\eta \quad (45)$$

$$\sigma_{z(1)}=\int_{-1}^{z}\left\{\mathbf{L}_8^{\mathrm{T}}\left[\mathbf{u}_1+\eta\boldsymbol{\phi}_1-\frac{\lambda\eta^3}{3}(\boldsymbol{\phi}_1+\mathbf{D}w_1)\right]+\tilde{l}_{63}w_1\right\}\mathrm{d}\eta$$
$$+\int_{-1}^{z}(z-\eta)\mathbf{D}^{\mathrm{T}}\left\{\mathbf{L}_7\left[\mathbf{u}_1+\eta\boldsymbol{\phi}_1-\frac{\lambda\eta^3}{3}(\boldsymbol{\phi}_1+\mathbf{D}w_1)\right]\right.$$
$$\left.+\mathbf{L}_8w_1\right\}\mathrm{d}\eta-f_{31}(x,y,z,\tau_0,\tau_1,\ldots)+\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z}\rho_2w_1\,\mathrm{d}\eta$$
$$-\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z}\rho_1(z-\eta)\mathbf{D}^{\mathrm{T}}\left[\mathbf{u}_1+\eta\boldsymbol{\phi}_1-\frac{\lambda\eta^3}{3}(\boldsymbol{\phi}_1+\mathbf{D}w_1)\right]\mathrm{d}\eta \quad (46)$$

where

$$\mathbf{u}_1=\{u_1,v_1\}^{\mathrm{T}},\quad \boldsymbol{\phi}_1=\{\phi_{x1},\phi_{y1}\}^{\mathrm{T}},\quad \varphi_{31}=-\int_0^z(\mathbf{L}_1\mathbf{u}_{(0)}+\tilde{l}_{33}w_{(0)})\,\mathrm{d}\eta$$

$$\boldsymbol{\varphi}_1=\{\varphi_{11},\varphi_{21}\}^{\mathrm{T}}=\int_0^z(\mathbf{L}_2\mathbf{u}_{(0)}+\mathbf{S}\hat{\boldsymbol{\sigma}}_{s(0)}-\mathbf{D}\varphi_{31})\,\mathrm{d}\eta$$

$$\mathbf{f}_1=\{f_{11},f_{21}\}^{\mathrm{T}}=-\int_{-1}^{z}(\mathbf{L}_7\boldsymbol{\varphi}_1+\mathbf{L}_8\varphi_{31}+\mathbf{L}_9\boldsymbol{\sigma}_{s(0)}+\mathbf{L}_{10}\sigma_{z(0)})\,\mathrm{d}\eta+\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z}\rho_1\boldsymbol{\varphi}_1\,\mathrm{d}\eta$$
$$+\frac{\partial^2}{\partial\tau_0\,\partial\tau_1}\int_{-1}^{z}2\rho_1\left[\mathbf{u}_0+\eta\boldsymbol{\phi}_0-\lambda\frac{\eta^3}{3}(\boldsymbol{\phi}_0+\mathbf{D}w_0)\right]\mathrm{d}\eta$$

$$f_{31}=-\int_{-1}^{z}(\mathbf{L}_8^{\mathrm{T}}\boldsymbol{\varphi}_1+\tilde{l}_{63}\varphi_{31}-\mathbf{D}^{\mathrm{T}}\mathbf{f}_1-\mathbf{L}_{12}\boldsymbol{\sigma}_{s(0)}-\tilde{l}_{64}\sigma_{z(0)})\,\mathrm{d}\eta$$
$$-\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z}\rho_2\varphi_{31}\,\mathrm{d}\eta-\frac{\partial^2}{\partial\tau_0\,\partial\tau_1}\int_{-1}^{z}2\rho_2w_0\,\mathrm{d}\eta.$$

w_1 and $\mathbf{u}_1$ are corrections to the midsurface displacements since $w_{(1)}=w_1$ and $u_{(1)}=u_1$ on $z=0$. Imposing the associated lateral boundary conditions (33)–(34) on eqns (45)–(46) and substituting eqns (43)–(44) into eqn (29), we arrive again at the FSDT and HSDT equations with nonhomogeneous terms known from the leading-order solution. The equations are

$$\mathbf{K}\boldsymbol{\Delta}_1=\mathbf{M}\frac{\partial^2\boldsymbol{\Delta}_1}{\partial\tau_0^2}+\mathbf{F}_1 \quad (47)$$

where

$$\boldsymbol{\Delta}_1=\{u_1,v_1,w_1,\phi_{x(1)},\phi_{y(1)}\}^{\mathrm{T}},\quad \mathbf{F}_1=\{h_{11},h_{21},h_{31},h_{41},h_{51}\}^{\mathrm{T}}$$
$$h_{11}=f_{11},\quad h_{21}=f_{21},\quad h_{31}=f_{31}-\frac{\partial g_{11}}{\partial x}-\frac{\partial g_{21}}{\partial y}-\frac{\lambda}{3}\left(\frac{\partial f_{11}}{\partial x}+\frac{\partial f_{21}}{\partial y}\right)$$

$$h_{41} = (1-\lambda/3)f_{11} - g_{11}, \quad h_{51} = (1-\lambda/3)f_{21} - g_{21}, \quad g_{j1} = \int_{-1}^{z} (1-\lambda z^2) f_{j1}\, d\eta \quad (j = 1, 2).$$

The asymptotic analysis can be continued to higher-orders in a similar way.

4. SOLVABILITY CONDITIONS

To ensure a uniform expansion of the asymptotic analysis, it is necessary to investigate the solvability conditions (Nayfeh, 1981) under which the higher-order equations possess solutions that are bounded and free from secular terms. Let us consider the free vibration characteristics and derive the solvability conditions for the higher-order equations. For the free vibration problem, we may simply set $\tilde{q} = 0$, $\mathbf{\Delta}_k = \tilde{\mathbf{\Delta}}_k \cos(\omega\tau_0 - \vartheta)$ in the equations, where ω is the circular frequency, ϑ the phase angle, and $\tilde{\mathbf{\Delta}}_k$ is the modal displacements. To simplify the notation, we express the resulting equations as:

$$L_j(u_k, v_k, w_k, \phi_{x(k)}, \phi_{y(k)}) = h_{jk} \quad (j = 1\text{–}5). \tag{48}$$

Multiplying eqn (48), respectively, by the adjoint functions φ_1, φ_2, φ_3, φ_4 and φ_5 to be determined later on, integrating and adding them up, we have

$$\sum_{j=1}^{5} \int_{\Omega} \varphi_j [L_j(u_k, v_k, w_k, \phi_{x(k)}, \phi_{y(k)}) - h_{jk}]\, d\Omega = 0. \tag{49}$$

The differential operation in eqn (49) can be transferred from u_k, v_k, w_k, $\phi_{x(k)}$, $\phi_{y(k)}$ to φ_1, φ_2, φ_3, φ_4, φ_5 by applying Green's theorem. After a lengthy, but straightforward, manipulation, we arrive at:

$$\begin{aligned}
&\int_{\Omega} u_k L_1(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5)\, d\Omega + \int_{\Omega} v_k L_2(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5)\, d\Omega \\
&\quad + \int_{\Omega} w_k L_3(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5)\, d\Omega + \int_{\Omega} \phi_{x(k)} L_4(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5)\, d\Omega \\
&\quad + \int_{\Omega} \phi_{y(k)} L_5(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5)\, d\Omega \\
&\quad - \int_{\Omega} [\varphi_1 h_{1k} + \varphi_2 h_{2k} + \varphi_3 h_{3k} + \varphi_4 h_{4k} + \varphi_5 h_{5k}]\, d\Omega \\
&\quad + \text{boundary integral terms} = 0.
\end{aligned} \tag{50}$$

If we choose, $\varphi_1 = u_0$, $\varphi_2 = v_0$, $\varphi_3 = -w_0$, $\varphi_4 = \phi_{x(0)}$, $\varphi_5 = \phi_{y(0)}$ as the adjoint functions, the first five domain integrals in eqn (50) vanish. Moreover, with this choice the boundary integral terms in eqn (50) are reduced to:

$$\begin{aligned}
&\int_{\Gamma} [u_{nk} N_{n(0)} - u_{n0} N_{n(k)}]\, d\Gamma + \int_{\Gamma} [u_{sk} N_{s(0)} - u_{s0} N_{s(k)}]\, d\Gamma + \int_{\Gamma} [w_k Q_{n(0)} - w_0 Q_{n(k)}]\, d\Gamma \\
&\qquad + \int_{\Gamma} [\phi_{n(k)} M_{n(0)} - \phi_{n(0)} M_{n(k)}]\, d\Gamma + \int_{\Gamma} [\phi_{s(k)} M_{s(0)} - \phi_{s(0)} M_{s(k)}]\, d\Gamma
\end{aligned} \tag{51}$$

where Γ denotes the edge boundary of the domain Ω; subscripts n and s denote the normal and tangential directions, respectively; $u_{nk} = u_k n_x + v_k n_y$, $u_{sk} = -u_k n_y + v_k n_x$, n_x and n_y are the components of the outward normal at a point on Γ:

$$N_{n(k)} = N_{x(k)}n_x^2 + N_{y(k)}n_y^2 + (N_{xy(k)} + N_{yx(k)})n_x n_y$$

$$N_{s(k)} = (N_{y(k)} - N_{x(k)})n_x n_y + N_{yx(k)}n_x^2 - N_{xy(k)}n_y^2$$

$$Q_{n(k)} = [Q_{x(k)} + \lambda(\tfrac{2}{3}P_{x(k),x} + \tfrac{2}{3}P_{xy(k),y} - R_{x(k)})]n_x + [Q_{y(k)} + \lambda(\tfrac{2}{3}P_{yx(k),x} + \tfrac{2}{3}P_{y(k),y} - R_{y(k)})]n_y$$

$$P_{n(k)} = P_{x(k)}n_x^2 + P_{y(k)}n_y^2 + (P_{xy(k)} + P_{yx(k)})n_x n_y$$

$$P_{s(k)} = (P_{y(k)} - P_{x(k)})n_x n_y + P_{yx(k)}n_x^2 - P_{xy(k)}n_y^2$$

$$M_{n(k)} = \left(M_{x(k)} - \frac{\lambda}{3}P_{x(k)}\right)n_x^2 + \left(M_{y(k)} - \frac{\lambda}{3}P_{y(k)}\right)n_y^2 + \left[(M_{xy(k)} + M_{yx(k)}) - \frac{\lambda}{3}(P_{xy(k)} + P_{yx(k)})\right]n_y^2$$

$$M_{s(k)} = \left[(M_{y(k)} - M_{x(k)}) - \frac{\lambda}{3}(P_{y(k)} - P_{x(k)})\right]n_x n_y + \left(M_{yx(k)} - \frac{\lambda}{3}P_{yx(k)}\right)n_x^2 + \left(M_{xy(k)} - \frac{\lambda}{3}P_{xy(k)}\right)n_y^2$$

$$(N_{x(k)}, M_{x(k)}, P_{x(k)}) = \int_{-1}^{1} \gamma_\beta \sigma_{x(k)}(1, z, z^3)\,dz$$

$$(N_{xy(k)}, M_{xy(k)}, P_{xy(k)}) = \int_{-1}^{1} \gamma_\beta \tau_{xy(k)}(1, z, z^3)\,dz$$

$$(N_{yx(k)}, M_{yx(k)}, P_{yx(k)}) = \int_{-1}^{1} \gamma_\alpha \tau_{xy(k)}(1, z, z^3)\,dz$$

$$(N_{y(k)}, M_{y(k)}, P_{y(k)}) = \int_{-1}^{1} \gamma_\alpha \sigma_{y(k)}(1, z, z^3)\,dz$$

$$(Q_{x(k)}, R_{x(k)}) = \int_{-1}^{1} \gamma_\beta \tau_{xz(k)}(1, z^2)\,dz$$

$$(Q_{y(k)}, R_{y(k)}) = \int_{-1}^{1} \gamma_\alpha \tau_{yz(k)}(1, z^2)\,dz.$$

The admissible edge conditions associated with ε^{2k}-order are derived from the boundary integral terms expressed in eqn (51), which require specification of the following:

$$u_n \text{ or } N_{n(k)}, \quad u_s \text{ or } N_{s(k)}, \quad w_k \text{ or } Q_{n(k)}, \quad \phi_{x(k)} \text{ or } M_{n(k)}, \quad \phi_{y(k)} \text{ or } M_{s(k)}. \tag{52}$$

It follows from eqn (50) that the solvability conditions are

$$u_0 h_{1k} + v_0 h_{2k} - w_0 h_{3k} + \phi_{x(0)} h_{4k} + \phi_{y(0)} h_{5k} = 0. \tag{53}$$

At the leading-order level this condition is identically satisfied. At subsequent levels, eqn (53) imposes an additional equation to be satisfied along with the higher-order equations. With h_{1k}, h_{2k}, h_{3k}, h_{4k} and h_{5k} as yet unknown functions of $\tau_1, \tau_2, \ldots$, the dependence of the field variables upon the scales $\tau_1, \tau_2, \ldots$ can be determined.

5. APPLICATIONS TO BENCHMARK PROBLEMS

The problem of simply-supported, doubly curved cross-ply shells is solved. The elastic constants of the cross-ply laminated shells are

$$(c_{16})_i = (c_{26})_i = (c_{36})_i = (c_{45})_i = 0. \tag{54}$$

The edge conditions are of a shear diaphragm type specified by

Order ε^{2k} $(k = 0, 1, 2\ldots)$:

$$v_k = w_k = \phi_{yk} = N_{x(k)} = M_{x(k)} = 0 \quad \text{on } \alpha = 0 \quad \text{and} \quad \alpha = a_\alpha \tag{55}$$

$$u_k = w_k = \phi_{xk} = N_{y(k)} = M_{y(k)} = 0 \quad \text{on } \beta = 0 \quad \text{and} \quad \beta = a_\beta. \tag{56}$$

The solutions to eqn (42) can be determined by letting

$$u_0 = U_0 \cos \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \vartheta) \tag{57}$$

$$v_0 = V_0 \sin \tilde{m}x \cos \tilde{n}y \cos(\omega\tau_0 - \vartheta) \tag{58}$$

$$w_0 = W_0 \sin \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \vartheta) \tag{59}$$

$$\phi_{x(0)} = \Phi_{x(0)} \cos \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \vartheta) \tag{60}$$

$$\phi_{y(0)} = \Phi_{y(0)} \sin \tilde{m}x \cos \tilde{n}y \cos(\omega\tau_0 - \vartheta) \tag{61}$$

in which $\tilde{m} = m\pi\sqrt{Rh}/a_\alpha$, $\tilde{n} = n\pi\sqrt{Rh}/a_\beta$ $(m, n = 1, 2, 3, \ldots)$, ω denotes the circular frequency of the motion. The phase angle ϑ is a function of $\tau_1, \tau_2, \ldots$, but not τ_0.

Substitution of eqns (57)–(61) into eqn (42) gives

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \end{bmatrix} \begin{Bmatrix} U_0 \\ V_0 \\ W_0 \\ \Phi_{x(0)} \\ \Phi_{y(0)} \end{Bmatrix} = \omega^2 \begin{bmatrix} m_{11} & 0 & m_{13} & m_{14} & 0 \\ 0 & m_{22} & m_{23} & 0 & m_{25} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} \\ m_{41} & 0 & m_{43} & m_{44} & 0 \\ 0 & m_{52} & m_{53} & 0 & m_{55} \end{bmatrix} \begin{Bmatrix} U_0 \\ V_0 \\ W_0 \\ \Phi_{x(0)} \\ \Phi_{y(0)} \end{Bmatrix} \tag{62}$$

where the expressions for k_{ij} and m_{ij} are given in the Appendix.

Equation (62) is an eigenvalue problem in which the mass and stiffness matrices are real and symmetric, hence ω_i must be real. When the coefficient matrices are positive definite, ω_i are positive. Nontrivial solution of eqn (62) exists for

$$\begin{vmatrix} k_{11}-\omega^2 m_{11} & k_{12} & k_{13}-\omega^2 m_{13} & k_{14}-\omega^2 m_{14} & k_{15} \\ k_{21} & k_{22}-\omega^2 m_{22} & k_{23}-\omega^2 m_{23} & k_{24} & k_{25}-\omega^2 m_{25} \\ k_{13}-\omega^2 m_{13} & k_{23}-\omega^2 m_{23} & k_{33}-\omega^2 m_{33} & k_{34}-\omega^2 m_{34} & k_{35}-\omega^2 m_{35} \\ k_{14}-\omega^2 m_{14} & k_{24} & k_{34}-\omega^2 m_{34} & k_{44}-\omega^2 m_{44} & k_{45} \\ k_{15} & k_{25}-\omega^2 m_{25} & k_{35}-\omega^2 m_{35} & k_{45} & k_{55}-\omega^2 m_{55} \end{vmatrix} = 0. \tag{63}$$

from which five eigenvalues ω_i $(i = 1\text{–}5)$ associated with the leading-order natural frequencies for a specific set of m and n can be determined.

To determine a unique solution, the modal displacements will be normalized by imposing

$$\begin{bmatrix} U_0+\varepsilon^2 U_1+\varepsilon^4 U_2+\cdots \\ V_0+\varepsilon^2 V_1+\varepsilon^4 V_2+\cdots \\ W_0+\varepsilon^2 W_1+\varepsilon^4 W_2+\cdots \\ \Phi_{x(0)}+\varepsilon^2\Phi_{x(1)}+\varepsilon^4\Phi_{x(2)}+\cdots \\ \Phi_{y(0)}+\varepsilon^2\Phi_{y(1)}+\varepsilon^4\Phi_{y(2)}+\cdots \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} U_0+\varepsilon^2 U_1+\varepsilon^4 U_2+\cdots \\ V_0+\varepsilon^2 V_1+\varepsilon^4 V_2+\cdots \\ W_0+\varepsilon^2 W_1+\varepsilon^4 W_2+\cdots \\ \Phi_{x(0)}+\varepsilon^2\Phi_{x(1)}+\varepsilon^4\Phi_{x(2)}+\cdots \\ \Phi_{y(0)}+\varepsilon^2\Phi_{y(1)}+\varepsilon^4\Phi_{y(2)}+\cdots \end{bmatrix} = 1. \tag{64}$$

The normalization conditions at each level are

ε^0-order:

$$U_0^2+V_0^2+W_0^2+\Phi_{x(0)}^2+\Phi_{y(0)}^2 = 1 \tag{65}$$

ε^2-order:

$$\begin{aligned} U_0^2+V_0^2+W_0^2+\Phi_{x(0)}^2+\Phi_{y(0)}^2 &= 1, \\ U_0U_1+V_0V_1+W_0W_1+\Phi_{x(0)}\Phi_{x(1)}+\Phi_{y(0)}\Phi_{y(1)} &= 0 \end{aligned} \tag{66}$$

ε^4-order:

$$\begin{aligned} U_0^2+V_0^2+W_0^2+\Phi_{x(0)}^2+\Phi_{y(0)}^2 &= 1, \\ U_0U_1+V_0V_1+W_0W_1+\Phi_{x(0)}\Phi_{x(1)}+\Phi_{y(0)}\Phi_{y(1)} &= 0, \\ U_1^2+2U_0U_2+V_1^2+2V_0V_2+W_1^2+2W_0W_2 & \\ +\Phi_{x(1)}^2+2\Phi_{x(0)}\Phi_{x(2)}+\Phi_{y(1)}^2+2\Phi_{y(0)}\Phi_{y(2)} &= 0,\ldots. \end{aligned} \tag{67}$$

For later use, we denote the normalized eigenvectors corresponding to ω_i ($i = 1$–5) as $[U_0^{(i)}\ \ V_0^{(i)}\ \ W_0^{(i)}\ \ \Phi_{x(0)}^{(i)}\ \ \Phi_{y(0)}^{(i)}]^{\mathrm{T}}$. The expressions for the modal stresses and displacements are given in the Appendix.

Carrying on the solution to higher-orders, we obtain the nonhomogeneous terms in the ε^2-order equations as

$$h_{11} = \left(\hat{h}_{11}\frac{\partial\vartheta_i}{\partial\tau_1}+\tilde{h}_{11}\right)\cos\tilde{m}x\sin\tilde{n}y\cos(\omega_i\tau_0-\vartheta_i) \tag{68}$$

$$h_{21} = \left(\hat{h}_{21}\frac{\partial\vartheta_i}{\partial\tau_1}+\tilde{h}_{21}\right)\sin\tilde{m}x\cos\tilde{n}y\cos(\omega_i\tau_0-\vartheta_i) \tag{69}$$

$$h_{31} = \left(\hat{h}_{31}\frac{\partial\vartheta_i}{\partial\tau_1}+\tilde{h}_{31}\right)\sin\tilde{m}x\sin\tilde{n}y\cos(\omega_i\tau_0-\vartheta_i) \tag{70}$$

$$h_{41} = \left(\hat{h}_{41}\frac{\partial\vartheta_i}{\partial\tau_1}+\tilde{h}_{41}\right)\cos\tilde{m}x\sin\tilde{n}y\cos(\omega_i\tau_0-\vartheta_i) \tag{71}$$

$$h_{51} = \left(\hat{h}_{51}\frac{\partial\vartheta_i}{\partial\tau_1}+\tilde{h}_{51}\right)\sin\tilde{m}x\cos\tilde{n}y\cos(\omega_i\tau_0-\vartheta_i) \tag{72}$$

where $\hat{h}_{ij}$ $\tilde{h}_{ij}$ are given in the Appendix.

In view of the similarity of the governing equations at the ε^2-order to those at the leading order, the ε^2-order solution can be determined again by letting

$$u_1 = U_1 \cos\tilde{m}x \sin\tilde{n}y \cos(\omega_i\tau_0-\vartheta_i) \tag{73}$$

$$v_1 = V_1 \sin\tilde{m}x \cos\tilde{n}y \cos(\omega_i\tau_0-\vartheta_i) \tag{74}$$

$$w_1 = W_1 \sin\tilde{m}x \sin\tilde{n}y \cos(\omega_i\tau_0-\vartheta_i) \tag{75}$$

$$\phi_{x(1)} = \Phi_{x(1)} \cos\tilde{m}x \sin\tilde{n}y \cos(\omega_i\tau_0-\vartheta_i) \tag{76}$$

$$\phi_{y(1)} = \Phi_{y(1)} \sin\tilde{m}x \cos\tilde{n}y \cos(\omega_i\tau_0-\vartheta_i). \tag{77}$$

Substituting eqns (73)–(77) into eqn (47) gives

$$\begin{bmatrix} k_{11}-\omega_i^2 m_{11} & k_{12} & k_{13}-\omega_i^2 m_{13} & k_{14}-\omega_i^2 m_{14} & k_{15} \\ k_{12} & k_{22}-\omega_i^2 m_{22} & k_{23}-\omega_i^2 m_{23} & k_{24} & k_{25}-\omega_i^2 m_{25} \\ k_{13}-\omega_i^2 m_{13} & k_{32}-\omega_i^2 m_{33} & k_{33}-\omega_i^2 m_{33} & k_{34}-\omega_i^2 m_{34} & k_{35}-\omega_i^2 m_{35} \\ k_{14}-\omega_i^2 m_{14} & k_{24} & k_{34}-\omega_i^2 m_{34} & k_{44}-\omega_i^2 m_{44} & k_{45} \\ k_{15} & k_{25}-\omega_i^2 m_{25} & k_{35}-\omega_i^2 m_{35} & k_{45} & k_{55}-\omega_i^2 m_{55} \end{bmatrix} \begin{Bmatrix} U_1 \\ V_1 \\ W_1 \\ \Phi_{x(1)} \\ \Phi_{y(1)} \end{Bmatrix}$$

$$= \begin{Bmatrix} \hat{h}_{11}(1)\dfrac{\partial\vartheta_i}{\partial\tau_1} + \tilde{h}_{11}(1) \\ \hat{h}_{21}(1)\dfrac{\partial\vartheta_i}{\partial\tau_1} + \tilde{h}_{21}(1) \\ \hat{h}_{31}(1)\dfrac{\partial\vartheta_i}{\partial\tau_1} + \tilde{h}_{31}(1) \\ \hat{h}_{41}(1)\dfrac{\partial\vartheta_i}{\partial\tau_1} + \tilde{h}_{41}(1) \\ \hat{h}_{51}(1)\dfrac{\partial\vartheta_i}{\partial\tau_1} + \tilde{h}_{51}(1) \end{Bmatrix}. \tag{78}$$

Equation (78) is solvable if and only if the solvability condition (53) is satisfied from which the dependence of ϑ_i upon τ_1 can be determined as

$$\vartheta_i = -\mu_i\tau_1 + \tilde{\vartheta}_i(\tau_2, \tau_3, \ldots) \tag{79}$$

where

$$\mu_i = \frac{U_0\tilde{h}_{11}(1)+V_0\tilde{h}_{21}(1)-W_0\tilde{h}_{31}(1)+\Phi_{x(0)}\tilde{h}_{41}+\Phi_{y(0)}\tilde{h}_{51}}{U_0\hat{h}_{11}(1)+V_0\hat{h}_{21}(1)-W_0\hat{h}_{31}(1)+\Phi_{x(0)}\hat{h}_{41}+\Phi_{y(0)}\hat{h}_{51}}$$

and $\tilde{\vartheta}_i$ are integration functions of the scales τ_2, τ_3, ..., which can be determined at the next-order level.

With eqn (79) and the relation $\tau_1 = \varepsilon^2\tau_0 = h\tau_0/R$, the time functions of the field variables are expressed in terms of $\cos[(\omega_i+\mu_i h/R)\tau_0-\tilde{\vartheta}_i]$. Consequently, the natural frequencies at the ε^2-order level have been modified to

$$\omega_i+\mu_i\frac{h}{R} \quad (i = 1,2,3,4,5). \tag{80}$$

Substituting eqn (80) into eqn (78), we obtain:

$$\begin{bmatrix} k_{11}-\omega_i^2 m_{11} & k_{12} & k_{13}-\omega_i^2 m_{13} & k_{14}-\omega_i^2 m_{14} & k_{15} \\ k_{12} & k_{22}-\omega_i^2 m_{22} & k_{23}-\omega_i^2 m_{23} & k_{24} & k_{25}-\omega_i^2 m_{25} \\ k_{13}-\omega_i^2 m_{13} & k_{23}-\omega_i^2 m_{23} & k_{33}-\omega_i^2 m_{33} & k_{34}-\omega_i^2 m_{34} & k_{35}-\omega_i^2 m_{35} \\ k_{14}-\omega_i^2 m_{14} & k_{24} & k_{34}-\omega_i^2 m_{34} & k_{44}-\omega_i^2 m_{44} & k_{45} \\ k_{15} & k_{25}-\omega_i^2 m_{25} & k_{35}-\omega_i^2 m_{35} & k_{45} & k_{55}-\omega_i^2 m_{55} \end{bmatrix} \begin{Bmatrix} U_1 \\ V_1 \\ W_1 \\ \Phi_{x(1)} \\ \Phi_{y(1)} \end{Bmatrix}$$
$$= \begin{Bmatrix} -\mu_i \hat{h}_{11}(1)+\tilde{h}_{11}(1) \\ -\mu_i \hat{h}_{21}(1)+\tilde{h}_{21}(1) \\ -\mu_i \hat{h}_{31}(1)+\tilde{h}_{31}(1) \\ -\mu_i \hat{h}_{41}(1)+\tilde{h}_{41}(1) \\ -\mu_i \hat{h}_{51}(1)+\tilde{h}_{51}(1) \end{Bmatrix}. \quad (81)$$

U_1, V_1, W_1, $\Phi_{x(1)}$ and $\Phi_{y(1)}$ can be uniquely determined by solving eqn (81) with the normalization conditions (66). Once they are obtained, determination of the ε^2-order corrections of the stresses and displacements is a straightforward matter.

6. ILLUSTRATIVE EXAMPLES

6.1. *Cylindrical and spherical laminated shells*

The formulation is applicable to multilayered cylindrical and spherical shells by letting $1/R_\alpha = 0$ and $R_\alpha = R_\beta$, respectively. The fundamental frequencies of the [0/90] laminated cylindrical and spherical shells with shear-diaphragm supports are examined. The geometry parameters of the shell are $a_\alpha/a_\beta = 1$; $R_\alpha/R_\beta = 1$; $R_\alpha/a_\alpha = 1, 5, 10$; and $2h/a_\alpha = 0.05, 0.10$ and 0.15. The material properties are $E_1/E_2 = 25$, $E_3 = E_2$, $G_{13} = G_{12} = 0.5E_2$, $G_{23} = 0.2E_2$, $v_{12} = 0.25$, $v_{31} = 0.03$, and $v_{23} = 0.4$. The normalized frequency Ω is defined as $\Omega = \omega a_\alpha \sqrt{\rho/E_2}$. Tables 1 and 2 give the natural frequencies based on CST, FSDT and HSDT-type asymptotic analyses for cylindrical and spherical shells. Comparisons with the available results (Ye and Soldatas, 1994; Bhimaraddi, 1984, 1991; Leissa and Qatu, 1991) are presented. The asymptotic solutions for the CST-type asymptotic model are convergent at the ε^4-order in the case of thin shells ($2h/a_\alpha = 0.05$), and at the ε^6-order in the case of thick shells ($2h/a_\alpha = 0.15$). For the refined asymptotic models, they are convergent at ε^2-order in the cases of thin shells ($2h/a_\alpha = 0.05$) and at the ε^4-order in the case of thick shells ($2h/a_\alpha = 0.15$). The refined asymptotic model speeds up the convergence.

6.2. *Doubly curved laminated shells*

The free vibration of the doubly curved laminated shell with shear–diaphragm supports is considered. The material properties are $E_1/E_2 = 25$; $E_3 = E_2$, $G_{13} = G_{12} = 0.5E_2$, $G_{23} = 0.2E_2$, $v_{12} = v_{13} = v_{23} = 0.25$. The normalized fundamental frequencies for [0/90/0] and [0/90/0/90] are presented in Table 3 for $R_\beta/a_\beta = 1, 5, 10$. Other geometry parameters are $a_\alpha/a_\beta = 1$, $R_\alpha/a_\alpha = 1, -1$ and $2h/a_\alpha = 0.05, 0.1$. Table 3 shows that the frequency decreases as R_β/a_β becomes large in the case of $R_\alpha/a_\alpha = 1$, which increases as R_β/a_β becomes large in the case of $R_\alpha/a_\alpha = -1$. The convergence is faster as the radii of curvature increase. The frequencies for $2h/a_\alpha = 0.05$ are smaller than those for $2h/a_\alpha = 0.1$. Table 4 presents the fundamental and higher frequencies for [0/90] laminated shells with $2h/a_\alpha = 0.1$. The convergence speed is greatly improved according to the refined asymptotic theories. The modal transverse stress distributions through the thickness of a [0/90/0] laminated shell based on various models are shown in Figs 2–7. The geometry parameters are $2h/a_\alpha = 0.05$, $a_\alpha/a_\beta = 1$, $R_\alpha/a_\alpha = 1$ and $R_\beta/a_\alpha = 10$. The transverse stresses are calculated by eqns (37) and (38), and (45) and (46). These equations are derived as the integral forms through the thickness direction. Accordingly, the continuity conditions for the interlaminar stresses are inherently satisfied. Indeed, as shown in Figs 2–7, the asymptotic solution yields continuous interlaminar stresses through the thickness and the traction boundary conditions at the lateral surfaces are satisfied exactly without special treatment.

Table 1. Fundamental frequencies ($\Omega = \omega a_\alpha \sqrt{\rho/E_2}$) of [0/90] cylindrical shells

$2h/a_\alpha$	R_β/a_β		Asymptotic theories CST-type	Asymptotic theories FSDT-type	Asymptotic theories HSDT-type	Ye–Soldatos (1994) State space	Bhimaraddi (1991) Layerwise	Bhimaraddi (1984) HSDT	Bhimaraddi (1984) FSDT ($k = \pi^2/12$)	Leissa–Qato (1991) CST
0.05	1	ε^0	0.80556	0.80016	0.79966	0.79316	0.78683	0.79993	0.79798	0.80580
		ε^2	0.79235	0.79256	0.79255					
		ε^4	0.79305	0.79290	0.79290					
		ε^6	0.79307	0.79292	0.79292					
0.05	5	ε^0	0.50233	0.49461	0.49418	0.49346	0.49167	0.49402	0.49091	0.50216
		ε^2	0.49307	0.49322	0.49322					
		ε^4	0.49332	0.49322	0.49322					
		ε^6	0.49331	0.49322	0.49322					
0.05	10	ε^0	0.48838	0.48050	0.48007	0.47959	0.47859	0.47997	0.47677	0.48827
		ε^2	0.47923	0.47942	0.47941					
		ε^4	0.47949	0.47942	0.47942					
		ε^6	0.47948	0.47942	0.47942					
0.10	1	ε^0	1.14432	1.09532	1.09256	1.06973	1.04085	1.09189	1.07475	1.14313
		ε^2	1.06117	1.06765	1.06756					
		ε^4	1.06901	1.06788	1.06789					
		ε^6	1.06875	1.06796	1.06796					
0.10	5	ε^0	0.96953	0.91284	0.91019	0.90616	0.90200	0.90953	0.88840	0.96870
		ε^2	0.89914	0.90532	0.90522					
		ε^4	0.90659	0.90553	0.90554					
		ε^6	0.90573	0.90552	0.90552					
0.10	10	ε^0	0.96120	0.90448	0.90184	0.89778	0.89564	0.90150	0.88026	0.96074
		ε^2	0.89067	0.89707	0.89698					
		ε^4	0.89829	0.89730	0.89731					
		ε^6	0.89740	0.89729	0.89728					
0.10	1	ε^0	1.54666	1.39096	1.38471	1.34537	1.29099	1.38174	1.33274	1.54124
		ε^2	1.30110	1.34059	1.33990					
		ε^4	1.34880	1.34103	1.34107					
		ε^6	1.34070	1.34103	1.34099					
0.15	5	ε^0	1.42657	1.26264	1.25665	1.24524	1.23849	1.25551	1.20020	1.42464
		ε^2	1.20437	1.24255	1.24178					
		ε^4	1.25336	1.24390	1.24407					
		ε^6	1.24210	1.24375	1.24372					
0.15	10	ε^0	1.41810	1.25514	1.24920	1.23707	1.23374	1.24875	1.19342	1.41709
		ε^2	1.19613	1.23470	1.23394					
		ε^4	1.24558	1.23605	1.23623					
		ε^6	1.23413	1.23591	1.23588					

Table 2. Fundamental frequencies ($\Omega = \omega a_\alpha \sqrt{\rho/E_2}$) of [0/90] spherical shells

$2h/a_\alpha$	R_β/a_β		Asymptotic theories CST-type	Asymptotic theories FSDT-type	Asymptotic theories HSDT-type	Bhimaraddi (1991) Layerwise	Bhimaraddi (1984) HSDT	Bhimaraddi (1984) FSDT ($k = \pi^2/12$)	Leissa–Qatu (1991) CST
0.05	1	ε^0	1.32988	1.32634	1.32583	1.29835	1.32595	1.32483	1.33000
		ε^2	1.30937	1.31047	1.31049				
		ε^4	1.31073	1.31111	1.31111				
		ε^6	1.31072	131114	1.31114				
0.05	5	ε^0	0.55246	0.54543	0.54500	0.54039	0.54500	0.54219	0.55247
		ε^2	0.54221	0.54254	0.54254				
		ε^4	0.54250	0.54254	0.54254				
		ε^6	0.54249	0.54254	0.54254				
0.05	10	ε^0	0.50149	0.49383	0.49341	0.49127	0.49341	0.49031	0.50149
		ε^2	0.49193	0.49218	0.49218				
		ε^4	0.49220	0.49219	0.49219				
		ε^6	0.49219	0.49219	0.49219				
0.10	1	ε^0	1.52339	1.49239	1.49023	1.39974	1.49075	1.48008	1.52391
		ε^2	1.42927	1.43995	1.44004				
		ε^4	1.44079	1.44099	1.44106				
		ε^6	1.43995	1.44129	1.44128				
0.10	5	ε^0	0.99032	0.93613	0.93359	0.92065	0.93361	0.91338	0.99034
		ε^2	0.91785	0.92471	0.92463				
		ε^4	0.92564	0.92487	0.92488				
		ε^6	0.92473	0.92486	0.92486				
0.10	10	ε^0	0.96519	0.90934	0.90679	0.89912	0.90679	0.88584	0.96519
		ε^2	0.89383	0.90048	0.90038				
		ε^4	0.90157	0.90068	0.90069				
		ε^6	0.90066	0.90067	0.90067				
0.15	1	ε^0	1.78806	1.68448	1.68011	1.51936	1.68141	1.64797	1.78940
		ε^2	1.54523	1.59015	1.58988				
		ε^4	1.59683	1.59051	1.59111				
		ε^6	1.58762	1.59155	1.59144				
0.15	5	ε^0	1.43115	1.27341	1.26792	1.24272	1.26797	1.21434	1.43120
		ε^2	1.20829	1.24722	1.24644				
		ε^4	1.25737	1.24833	1.24854				
		ε^6	1.24612	1.24823	1.24820				
0.15	10	ε^0	1.41638	1.25586	1.25032	1.23249	1.25034	1.19559	1.41639
		ε^2	1.19447	1.23314	1.23232				
		ε^4	1.24377	1.23440	1.23459				
		ε^6	1.23240	1.23428	1.23424				

Table 3. Fundamental frequencies ($\Omega = \omega a_\alpha \sqrt{\rho/E_2}$) of doubly curved shells

Laminates	$2h/a_\alpha$	R_α/a_α	R_β/a_β	CST-type asymptotic theory ε^0	ε^2	ε^4	ε^6	FSDT-type asymptotic theory ε^0	ε^2	ε^4	ε^6	HSDT-type asymptotic theory ε^0	ε^2	ε^4	ε^6
[0/90/0]	0.05	1	1	1.43394	1.36589	1.37919	1.37696	1.41613	1.37393	1.37733	1.37729	1.40943	1.37592	1.37715	1.37729
		1	5	1.06151	0.96748	0.98469	0.98203	1.03548	0.97822	0.98241	0.98241	1.02562	0.98088	0.98221	0.98240
		1	10	1.01554	0.91728	0.93493	0.93224	0.98826	0.92835	0.93262	0.93262	0.97792	0.93110	0.93241	0.93261
		−1	1	0.68854	0.54898	0.56644	0.56374	0.65121	0.56106	0.56415	0.56412	0.63676	0.56397	0.56397	0.56412
		−1	5	0.88662	0.77453	0.79311	0.79040	0.85540	0.78645	0.79075	0.79078	0.84350	0.78941	0.79055	0.79078
		−1	10	0.92767	0.82033	0.83868	0.83597	0.89777	0.83200	0.83633	0.83635	0.88640	0.83490	0.83613	0.83635
	0.10	1	1	1.86708	1.35909	1.71321	1.45705	1.69958	1.53898	1.57031	1.56439	1.65460	1.56139	1.56471	1.56548
		1	5	1.63341	1.01879	1.42641	1.14085	1.42276	1.22853	1.26585	1.25886	1.36467	1.25550	1.25932	1.26013
		1	10	1.60468	0.97825	1.39009	1.10251	1.38926	1.19047	1.22830	1.22124	1.32962	1.21785	1.22170	1.22252
		−1	1	1.37079	0.68561	1.09582	0.81100	1.13518	0.89889	0.93575	0.92838	1.06787	0.92650	0.92909	0.92970
		−1	5	1.52338	0.86618	1.28618	0.99499	1.29570	1.08346	1.12218	1.11497	1.23192	1.11173	1.11543	1.11628
		−1	10	1.54948	0.90164	1.31969	1.02931	1.32548	1.11765	1.15618	1.14901	1.26298	1.14568	1.14947	1.15031
[0/90/0/90]	0.05	1	1	1.41061	1.36273	1.36879	1.36833	1.40080	1.36609	1.36835	1.36840	1.39864	1.36668	1.36861	1.36840
		1	5	1.02195	0.96875	0.97501	0.97445	1.00789	0.97239	0.97452	0.97451	1.00464	0.97297	0.97448	0.97451
		1	10	0.97361	0.91835	0.92481	0.92424	0.95885	0.92209	0.92430	0.92429	0.95544	0.92267	0.92426	0.92429
		−1	1	0.64406	0.54111	0.54808	0.54743	0.62176	0.54606	0.54765	0.54761	0.61670	0.54698	0.54759	0.54761
		−1	5	0.85141	0.79329	0.79961	0.79908	0.83316	0.79710	0.79915	0.79915	0.82907	0.79775	0.79910	0.79915
		−1	10	0.89423	0.83964	0.84579	0.84527	0.87681	0.84332	0.84533	0.84532	0.87292	0.84394	0.84528	0.84532
	0.10	1	1	1.78793	1.47183	1.59893	1.54929	1.68750	1.54527	1.56579	1.56403	1.66851	1.55392	1.56411	1.56419
		1	5	1.52017	1.17610	1.31041	1.25619	1.39643	1.25297	1.27484	1.27188	1.37190	1.26154	1.27300	1.27220
		1	10	1.48752	1.13755	1.27330	1.21852	1.36118	1.21498	1.23737	1.23427	1.33600	1.22362	1.23548	1.23462
		−1	1	1.30109	0.84469	1.00322	0.93669	1.14327	0.93752	0.96201	0.95796	1.11387	0.94865	0.95947	0.95846
		−1	5	1.44672	1.06889	1.21552	1.15472	1.29841	1.15368	1.17624	1.17278	1.27062	1.16325	1.17404	1.17319
		−1	10	1.47250	1.10316	1.24807	1.18813	1.32732	1.18690	1.20908	1.20572	1.30012	1.19631	1.20692	1.20612

Table 4. Natural frequencies (Ω) of [0/90] doubly curved shells ($\Omega = \omega a_\alpha \sqrt{\rho/E_2}$)

R_α/a_α	R_β/a_β	(m, n)	CST-type asymptotic theory ε^0	ε^2	ε^4	ε^6	FSDT-type asymptotic theory ε^0	ε^2	ε^4	ε^6	HSDT-type asymptotic theory ε^0	ε^2	ε^4	ε^6
1	1	(1, 1)	1.52339	1.43057	1.44172	1.44092	1.49239	1.43983	1.44083	1.44116	1.49023	1.43996	1.44089	1.44115
		(1, 2)	2.92076	2.41370	2.55951	2.51017	2.62706	2.52347	2.52988	2.52915	2.62282	2.52217	2.53087	2.52897
		(1, 3)	5.54364	3.29967	4.77753	3.71317	4.36292	4.14216	4.19787	4.17793	4.36825	4.11439	4.21669	4.16838
		(2, 1)	2.89605	2.29307	2.52583	2.43243	2.56903	2.45674	2.45903	2.45951	2.56159	2.45135	2.46149	2.45889
		(2, 2)	3.82289	2.86916	3.18986	3.08285	3.25234	3.10866	3.11301	3.11288	3.23856	3.10342	3.11596	3.11220
		(2, 3)	6.12950	3.58199	5.09852	4.09351	4.72747	4.48541	4.53662	4.51797	4.72248	4.45522	4.55750	4.50730
		(3, 1)	5.49839	2.98722	4.84992	3.40348	4.25432	4.01649	4.04669	4.03379	4.25692	3.97066	4.07719	4.01697
		(3, 2)	6.11704	3.37134	5.19283	3.86895	4.66702	4.42367	4.44415	4.43502	4.65824	4.38125	4.47334	4.41883
		(3, 3)	7.82765	4.15516	6.27387	5.13680	5.79192	5.49048	5.54335	5.52219	5.78477	5.43471	5.58397	5.49937
1	5	(1, 1)	1.17217	1.07168	1.08356	1.08259	1.13377	1.08115	1.08242	1.08272	1.13169	1.08133	1.08248	1.08271
		(1, 2)	2.76615	2.27135	2.42899	2.37234	2.47070	2.38228	2.38987	2.38888	2.46672	2.37987	2.39081	2.38864
		(1, 3)	5.40565	3.22345	4.69272	3.62677	4.25861	4.04729	4.09637	4.07726	4.26372	4.01531	4.11605	4.06676
		(2, 1)	2.56499	1.90335	2.13094	2.04703	2.19684	2.06817	2.06918	2.06983	2.18996	2.06220	2.07155	2.06929
		(2, 2)	3.64325	2.69325	3.01772	2.90802	3.06831	2.92926	2.93573	2.93538	3.05460	2.92356	2.93824	2.93476
		(2, 3)	5.96963	3.46479	4.99086	3.95952	4.59567	4.35678	4.40243	4.38484	4.59002	4.32344	4.42347	4.37349
		(3, 1)	5.25740	2.66856	4.44541	3.14940	3.94266	3.67753	3.70638	3.69306	3.94628	3.62943	3.73657	3.67695
		(3, 2)	5.93988	3.25112	4.96564	3.77540	4.50632	4.25221	4.27791	4.26767	4.49806	4.21015	4.30531	4.25274
		(3, 3)	7.65177	4.05728	6.13469	5.00604	5.66289	5.35327	5.40552	5.38460	5.65515	5.29648	5.44416	5.36276
1	10	(1, 1)	1.12991	1.02712	1.03925	1.03828	1.09047	1.03669	1.03807	1.03837	1.08838	1.03687	1.03813	1.03836
		(1, 2)	2.74547	2.24854	2.40822	2.35087	2.44990	2.36035	2.36788	2.36689	2.44592	2.35779	2.36883	2.36664
		(1, 3)	5.38697	3.20821	4.67723	3.61209	4.24409	4.03093	4.07892	4.06007	4.24915	3.99842	4.09877	4.04946
		(2, 1)	2.53753	1.87499	2.10025	2.01788	2.16858	2.03843	2.03948	2.04014	2.16183	2.03246	2.04181	2.03962
		(2, 2)	3.62190	2.67072	2.99573	2.88572	3.04755	2.90656	2.91314	2.91277	3.03384	2.90080	2.91559	2.91217
		(2, 3)	5.94879	3.44572	4.97366	3.93993	4.57872	4.33731	4.38202	4.36469	4.57296	4.30357	4.40313	4.35328
		(3, 1)	5.23622	2.65799	4.41462	3.14163	3.92488	3.65792	3.68682	3.67351	3.92852	3.61004	3.71669	3.65763
		(3, 2)	5.91876	3.23990	4.93959	3.76550	4.49056	4.23358	4.25985	4.24946	4.48235	4.19169	4.28695	4.23474
		(3, 3)	7.62920	4.04263	6.11528	4.98775	5.64732	5.33432	5.38631	5.36549	5.63948	5.27744	5.42470	5.34380
1	20	(1, 1)	1.10925	1.00516	1.01741	1.01644	1.06929	1.01479	1.01620	1.01650	1.06720	1.01496	1.01626	1.01649
		(1, 2)	2.73505	2.23674	2.39747	2.33981	2.43942	2.34904	2.35653	2.35554	2.43544	2.34641	2.35749	2.35529
		(1, 3)	5.37753	3.20013	4.66909	3.60446	4.23672	4.02238	4.06982	4.05110	4.24174	3.98961	4.08976	4.04043
		(2, 1)	2.52528	1.86292	2.08689	2.00531	2.15643	2.02555	2.02664	2.02729	2.14974	2.01959	2.02894	2.02678
		(2, 2)	3.61136	2.65943	2.98473	2.87457	3.03739	2.89522	2.90184	2.90146	3.02369	2.88943	2.90427	2.90086
		(2, 3)	5.93831	3.43583	4.96476	3.92993	4.57022	4.32730	4.37152	4.35434	4.56439	4.29336	4.39267	4.34289
		(3, 1)	5.22652	2.65531	4.40114	3.13989	3.91799	3.65022	3.67917	3.66586	3.92165	3.60249	3.70885	3.65011
		(3, 2)	5.90831	2.23464	4.92679	3.76073	4.48307	4.22453	4.25107	4.24062	4.47489	4.18274	4.27801	4.22599
		(3, 3)	7.61788	4.03512	6.10541	4.97847	5.63959	5.32470	5.37654	5.35579	5.63169	5.26778	5.41482	5.33418

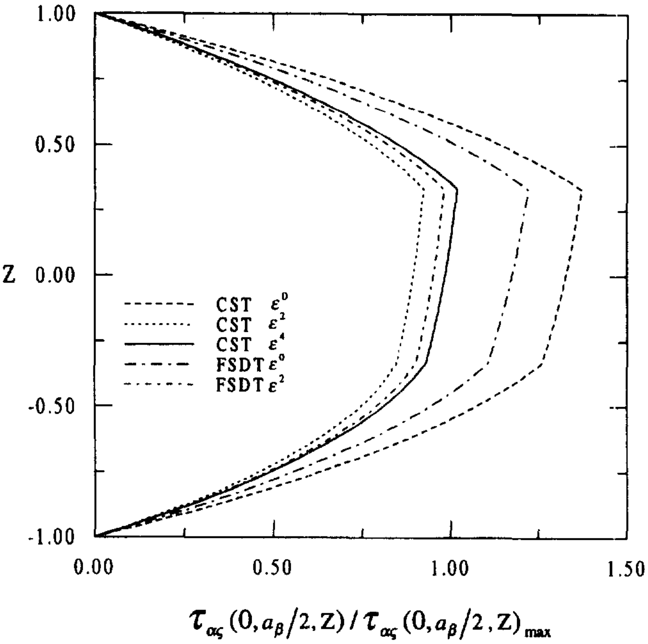

Fig. 2. The distribution of the modal transverse shear stress through the thickness of the [0/90/0] doubly curved shell.

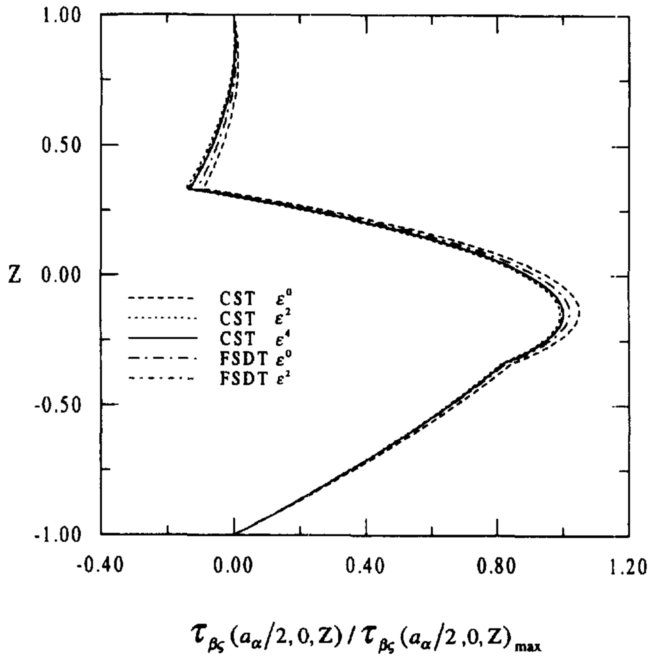

Fig. 3. The distribution of the modal transverse shear stress through the thickness of the [0/90/0] doubly curved shell.

7. CONCLUSIONS

A refined asymptotic theory leading to FSDT and HSDT equations at each level of approximation has been developed for dynamic analysis of the multilayered double curved shell with constant radii of curvature. As a result of bringing the transverse shear deformations to the stage at the leading order, the asymptotic solution converges to the correct solution rapidly. Applications to the benchmark problems show that considerable improvements can be achieved for thick laminated shells. Accurate results are obtained by performing only one step of the solution in the case of thin laminated shells ($2h/a_\alpha = 0.01$),

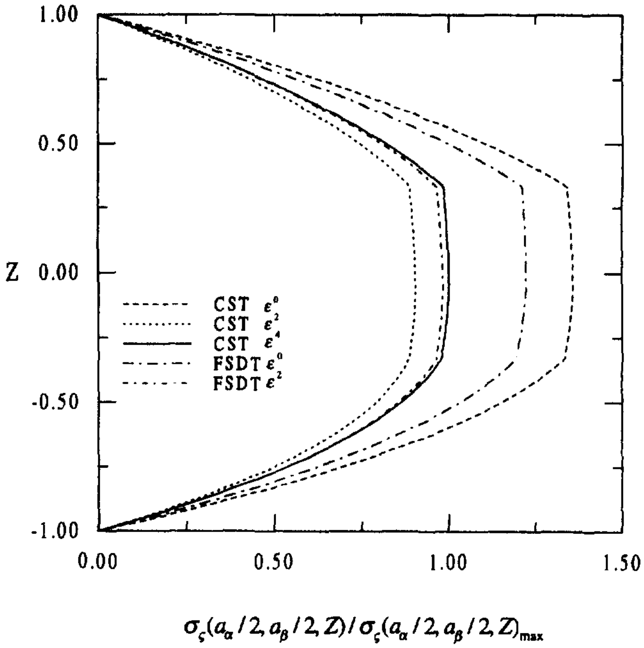

Fig. 4. The distribution of the modal transverse normal stress through the thickness of the [0/90/0] doubly curved shell.

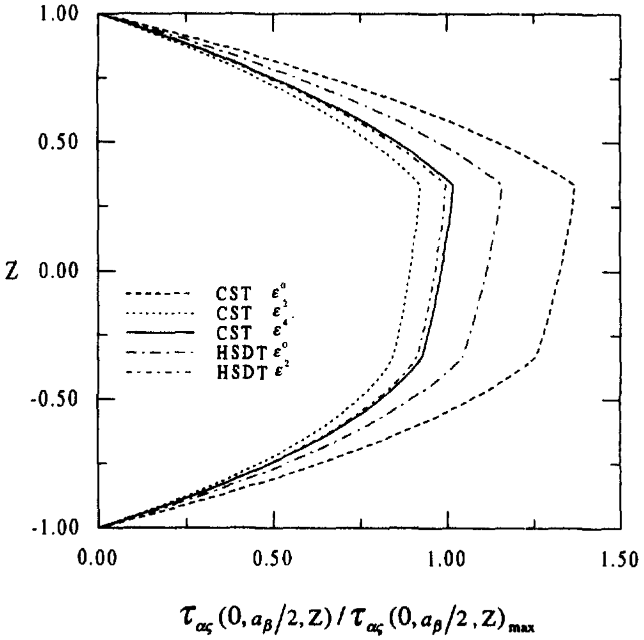

Fig. 5. The distribution of the modal transverse shear stress through the thickness of the [0/90/0] doubly curved shell.

and two steps in the case of thick shells ($2h/a_\alpha = 0.15$). After two more steps, the solutions based on CST, FSDT and HSDT-type models approach to the same result. With the refined models, results of similar accuracy can be obtained with at least one step less than that of the asymptotic model without refinement.

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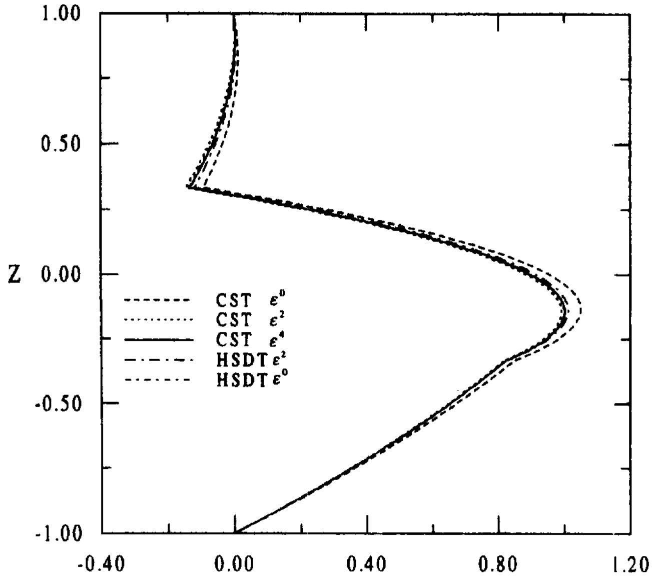
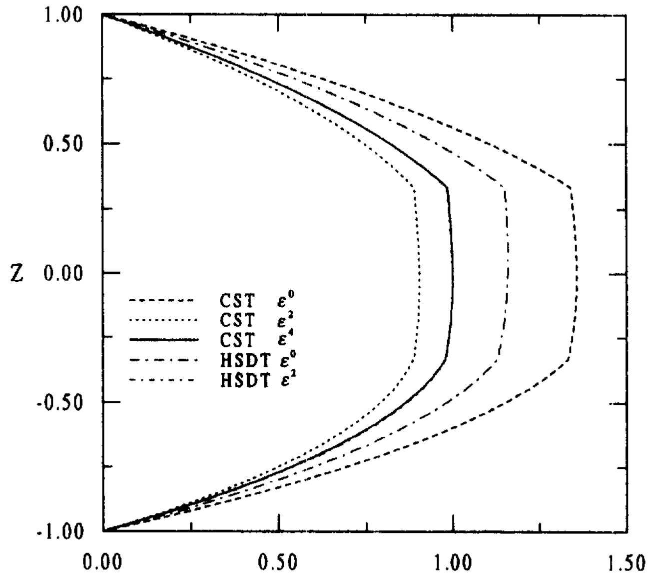


$\tau_{\beta\varsigma}(a_\alpha/2, 0, Z) / \tau_{\beta\varsigma}(a_\alpha/2, 0, Z)_{max}$

Fig. 6. The distribution of the modal transverse shear stress through the thickness of the [0/90/0] doubly curved shell.



$\sigma_\varsigma(a_\alpha/2, a_\beta/2, Z) / \sigma_\varsigma(a_\alpha/2, a_\beta/2, Z)_{max}$

Fig. 7. The distribution of the modal transverse normal stress through the thickness of the [0/90/0] doubly curved shell.

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APPENDIX

The expressions for k_{ij} and m_{ij} in eqn (62) are

$$k_{11} = \tilde{m}^2 \hat{A}_{11} + \tilde{n}^2 \bar{A}_{66}, \quad k_{12} = k_{21} = \tilde{m}\tilde{n}(\tilde{A}_{12} + \tilde{A}_{66})$$

$$k_{13} = k_{31} = -\tilde{m}\left(\frac{\hat{A}_{11}}{R_x} + \frac{\tilde{A}_{12}}{R_y}\right) - \frac{\lambda}{3}[\tilde{m}^3 \hat{E}_{11} + \tilde{m}\tilde{n}^2(\bar{E}_{66} + \tilde{E}_{12} + \tilde{E}_{66})]$$

$$k_{14} = k_{41} = \tilde{m}^2 \hat{B}_{11} + \tilde{n}^2 \bar{B}_{66} - \frac{\lambda}{3}(\tilde{m}^2 \hat{E}_{11} + \tilde{n}^2 \bar{E}_{66})$$

$$k_{15} = k_{51} = \tilde{m}\tilde{n}(\tilde{B}_{12} + \bar{B}_{66}) - \frac{\lambda}{3}\tilde{m}\tilde{n}(\tilde{E}_{12} + \tilde{E}_{66}), \quad k_{22} = \tilde{m}^2 \hat{A}_{66} + \tilde{n}^2 \bar{A}_{22}$$

$$k_{23} = k_{32} = -\tilde{n}\left(\frac{\tilde{A}_{12}}{R_x} + \frac{\bar{A}_{22}}{R_y}\right) - \frac{\lambda}{3}[\tilde{m}^2\tilde{n}(\tilde{E}_{12} + \tilde{E}_{66} + \hat{E}_{66}) + \tilde{n}^3 \bar{E}_{22}]$$

$$k_{24} = k_{42} = k_{15}, \quad k_{25} = k_{52} = \tilde{m}^2 \hat{B}_{66} + \tilde{n}^2 \bar{B}_{22} - \frac{\lambda}{3}(\tilde{m}^2 \hat{E}_{66} + \tilde{n}^2 \bar{E}_{22})$$

$$k_{33} = \left(\frac{\hat{A}_{11}}{R_x^2} + \frac{2\tilde{A}_{12}}{R_x R_y} + \frac{\bar{A}_{22}}{R_y^2}\right) + \tilde{m}^2 \tilde{A}_{55} + \tilde{n}^2 \tilde{A}_{44} + \lambda\left[\frac{2\tilde{m}^2}{3}\left(\frac{\hat{E}_{11}}{R_x} + \frac{\tilde{E}_{12}}{R_y}\right) + \frac{2\tilde{n}^2}{3}\left(\frac{\tilde{E}_{12}}{R_x} + \frac{\bar{E}_{22}}{R_y}\right) + \frac{\tilde{m}^4 \hat{H}_{11}}{9}\right.$$
$$\left. + \frac{\tilde{m}^2\tilde{n}^2}{9}(2\tilde{H}_{12} + \bar{H}_{66} + 2\tilde{H}_{66} + \hat{H}_{66}) + \frac{\tilde{n}^4 \bar{H}_{22}}{9} - \tilde{m}^2(2\tilde{D}_{55} - \tilde{F}_{55}) - \tilde{n}^2(2\tilde{D}_{44} - \tilde{F}_{44})\right]$$

$$k_{34} = k_{43} = -\tilde{m}\left(\frac{\hat{B}_{11}}{R_x} + \frac{\tilde{B}_{12}}{R_y}\right) + \tilde{m}\tilde{A}_{55} + \lambda\left[\tilde{m}\left(\frac{\hat{E}_{11}}{3R_x} + \frac{\tilde{E}_{12}}{3R_y}\right) - \frac{\tilde{m}^3}{3}\left(\hat{F}_{11} - \frac{\hat{H}_{11}}{3}\right)\right.$$
$$\left. - \frac{\tilde{m}\tilde{n}^2}{3}\left(\bar{F}_{66} - \frac{\bar{H}_{66}}{3} + \tilde{F}_{12} - \frac{\tilde{H}_{12}}{3} + \tilde{F}_{66} - \frac{\tilde{H}_{66}}{3}\right) - \tilde{m}(2\tilde{D}_{55} - \tilde{F}_{55})\right]$$

$$k_{35} = k_{53} = -\tilde{n}\left(\frac{\tilde{B}_{12}}{R_x} + \frac{\bar{B}_{22}}{R_y}\right) + \tilde{n}\tilde{A}_{44} + \lambda\left[\tilde{n}\left(\frac{\tilde{E}_{12}}{3R_x} + \frac{\bar{E}_{22}}{3R_y}\right) - \frac{\tilde{m}^2\tilde{n}}{3}\left(\tilde{F}_{12} - \frac{\tilde{H}_{12}}{3} + \tilde{F}_{66} - \frac{\tilde{H}_{66}}{3} + \hat{F}_{66} - \frac{\hat{H}_{66}}{3}\right)\right.$$
$$\left. - \frac{\tilde{n}^3}{3}\left(\bar{F}_{22} - \frac{\bar{H}_{22}}{3}\right) - \tilde{n}(2\tilde{D}_{44} - \tilde{F}_{44})\right]$$

$$k_{44} = \tilde{m}^2 \hat{D}_{11} + \tilde{n}^2 \bar{D}_{66} + \tilde{A}_{55} - \lambda\left[\tilde{m}^2\left(\frac{2\hat{F}_{11}}{3} - \frac{\hat{H}_{11}}{9}\right) + \tilde{n}^2\left(\frac{2\bar{F}_{66}}{3} - \frac{\bar{H}_{66}}{9}\right) + (2\tilde{D}_{55} - \tilde{F}_{55})\right]$$

$$k_{45} = k_{54} = \tilde{m}\tilde{n}(\tilde{D}_{12} + \tilde{D}_{66}) - \lambda\tilde{m}\tilde{n}(\tfrac{2}{3}\tilde{F}_{12} - \tfrac{1}{9}\tilde{H}_{12} + \tfrac{2}{3}\tilde{F}_{66} - \tfrac{1}{9}\tilde{H}_{66})$$

$$k_{55} = \tilde{m}^2 \hat{D}_{66} + \tilde{n}^2 \bar{D}_{22} + \tilde{A}_{44} - \lambda\left[\tilde{m}^2\left(\frac{2\hat{F}_{66}}{3} - \frac{\hat{H}_{66}}{9}\right) + \tilde{n}^2\left(\frac{2\bar{F}_{22}}{3} - \frac{\bar{H}_{22}}{9}\right) + (2\tilde{D}_{44} - \tilde{F}_{44})\right]$$

$$m_{11} = m_{22} = I_{10}, \quad m_{13} = m_{31} = -\lambda\frac{\tilde{m}I_{13}}{3}, \quad m_{14} = m_{41} = m_{25} = m_{52} = I_{11} - \lambda\frac{I_{13}}{3}$$

$$m_{23} = m_{32} = -\lambda\frac{\tilde{n}I_{13}}{3}, \quad m_{33} = I_{20} + \lambda\frac{(\tilde{m}^2 + \tilde{n}^2)I_{16}}{9}, \quad m_{34} = m_{43} = -\lambda\tilde{m}\left(\frac{I_{14}}{3} - \frac{I_{16}}{9}\right)$$

$$m_{35} = m_{53} = -\lambda\tilde{n}\left(\frac{I_{14}}{3} - \frac{I_{16}}{9}\right), \quad m_{44} = m_{55} = I_{12} - \lambda\left(\frac{2I_{14}}{3} - \frac{I_{16}}{9}\right)$$

$$m_{12} = m_{21} = m_{15} = m_{51} = m_{24} = m_{42} = m_{45} = m_{54} = 0. \qquad \text{(A1a–v)}$$

The ε^{2k}-order solution for the problem are

$$u_{(k)} = \tilde{u}_{(k)} \cos \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \vartheta) \tag{A2}$$

$$v_{(k)} = \tilde{v}_{(k)} \sin \tilde{m}x \cos \tilde{n}y \cos(\omega\tau_0 - \vartheta) \tag{A3}$$

$$w_{(k)} = \tilde{w}_{(k)} \sin \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \vartheta) \tag{A4}$$

$$\phi_{x(k)} = \tilde{\phi}_{x(k)} \cos \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \vartheta) \tag{A5}$$

$$\phi_{y(k)} = \tilde{\phi}_{y(k)} \sin \tilde{m}x \cos \tilde{n}y \cos(\omega\tau_0 - \vartheta) \tag{A6}$$

$$\sigma_{x(k)} = \tilde{\sigma}_{x(k)} \sin \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \vartheta) \tag{A7}$$

$$\sigma_{y(k)} = \tilde{\sigma}_{y(k)} \sin \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \vartheta) \tag{A8}$$

$$\sigma_{z(k)} = \tilde{\sigma}_{z(k)} \sin \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \vartheta) \tag{A9}$$

$$\tau_{xz(k)} = \tilde{\tau}_{xz(k)} \cos \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \vartheta) \tag{A10}$$

$$\tau_{yz(k)} = \tilde{\tau}_{yz(k)} \sin \tilde{m}x \cos \tilde{n}y \cos(\omega\tau_0 - \vartheta) \tag{A11}$$

$$\tau_{xy(k)} = \tilde{\tau}_{xy(k)} \cos \tilde{m}x \cos \tilde{n}y \cos(\omega\tau_0 - \vartheta) \tag{A12}$$

$$\hat{\tau}_{xz(k)} = \bar{\tau}_{xz(k)} \cos \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \vartheta) \tag{A13}$$

$$\hat{\tau}_{yz(k)} = \bar{\tau}_{yz(k)} \sin \tilde{m}x \cos \tilde{n}y \cos(\omega\tau_0 - \vartheta) \tag{A14}$$

where, at the ε^0-order level

$$\tilde{u}_{(0)} = U_0 + z\Phi_{x(0)} - \frac{\lambda z^3}{3}[\Phi_{x(0)} + z\tilde{m}W_0] \tag{A15}$$

$$\tilde{v}_{(0)} = V_0 + z\Phi_{y(0)} - \frac{\lambda z^3}{3}[\Phi_{y(0)} + \tilde{n}W_0] \tag{A16}$$

$$\tilde{w}_{(0)} = W_0 \tag{A17}$$

$$\tilde{\sigma}_{x(0)} = -\frac{\tilde{Q}_{11}\tilde{m}}{\gamma_\alpha}U_0 - \frac{\tilde{Q}_{12}\tilde{n}}{\gamma_\beta}V_0 + \left(\frac{\tilde{Q}_{11}}{\gamma_\alpha R_x} + \frac{\tilde{Q}_{12}}{\gamma_\beta R_y}\right)W_0 \tag{A18}$$

$$\tilde{\sigma}_{y(0)} = -\frac{\tilde{Q}_{12}\tilde{m}}{\gamma_\alpha}U_0 - \frac{\tilde{Q}_{22}\tilde{n}}{\gamma_\beta}V_0 + \left(\frac{\tilde{Q}_{12}}{\gamma_\alpha R_x} + \frac{\tilde{Q}_{22}}{\gamma_\beta R_y}\right)W_0 \tag{A19}$$

$$\tilde{\tau}_{xy(0)} = \frac{\tilde{Q}_{66}\tilde{n}}{\gamma_\beta}U_0 + \frac{\tilde{Q}_{66}\tilde{m}}{\gamma_\alpha}V_0 \tag{A20}$$

$$\tilde{\tau}_{xz(0)} = \int_{-1}^{z}\left[\left(\frac{\tilde{m}^2\tilde{Q}_{11}\gamma_\beta}{\gamma_\alpha} + \frac{\tilde{n}^2\tilde{Q}_{66}\gamma_\alpha}{\gamma_\beta}\right)\tilde{u}_{(0)} + \tilde{m}\tilde{n}(\tilde{Q}_{12}+\tilde{Q}_{66})\tilde{v}_{(0)} - \tilde{m}\left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x\gamma_\alpha} + \frac{\tilde{Q}_{12}}{R_y}\right)\tilde{w}_{(0)} - \rho_1\omega^2\tilde{u}_{(0)}\right]\mathrm{d}\eta \tag{A21}$$

$$\tilde{\tau}_{yz(0)} = \int_{-1}^{z}\left[\tilde{m}\tilde{n}(\tilde{Q}_{12}+\tilde{Q}_{66})\tilde{u}_{(0)} + \left(\frac{\tilde{m}^2\tilde{Q}_{66}\gamma_\beta}{\gamma_\alpha} + \frac{\tilde{n}^2\tilde{Q}_{22}\gamma_\alpha}{\gamma_\beta}\right)\tilde{v}_{(0)} - \tilde{n}\left(\frac{\tilde{Q}_{12}}{R_x} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y\gamma_\beta}\right)\tilde{w}_{(0)} - \rho_1\omega^2\tilde{v}_{(0)}\right]\mathrm{d}\eta \tag{A22}$$

$$\begin{aligned}\tilde{\sigma}_{z(0)} = \int_{-1}^{z}\Bigg[&-\tilde{m}\left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x\gamma_\alpha} + \frac{\tilde{Q}_{12}}{R_y}\right)\tilde{u}_{(0)} - \tilde{n}\left(\frac{\tilde{Q}_{12}}{R_x} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y\gamma_\beta}\right)\tilde{v}_{(0)} \\ &+ \left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x^2\gamma_\alpha} + \frac{2\tilde{Q}_{12}}{R_xR_y} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y^2\gamma_\beta}\right)\tilde{w}_{(0)} + \tilde{m}\tilde{\tau}_{xz(0)} + \tilde{n}\tilde{\tau}_{yz(0)} - \rho_2\omega^2\tilde{w}_{(0)}\Bigg]\mathrm{d}\eta.\end{aligned} \tag{A23}$$

The expressions of $\hat{f}_{i1}, \tilde{f}_{i1}, \hat{g}_{i1}, \tilde{g}_{i1}, \hat{h}_{i1}, \tilde{h}_{i1}$ and the relevant functions for the ε^2-order corrections are

$$\hat{h}_{31} = \hat{f}_{31} + \tilde{m}\hat{g}_{11} + \tilde{n}\hat{g}_{21} + \frac{\lambda}{3}(\tilde{m}\hat{f}_{11} + \tilde{n}\hat{f}_{21}), \quad \tilde{h}_{31} = \tilde{f}_{31} + \tilde{m}\tilde{g}_{11} + \tilde{n}\tilde{g}_{21} + \frac{\lambda}{3}(\tilde{m}\tilde{f}_{11} + \tilde{n}\tilde{f}_{21})$$

$$\hat{h}_{41} = \left(1 - \frac{\lambda}{3}\right)\hat{f}_{11} - \hat{g}_{11}, \quad \tilde{h}_{41} = \left(1 - \frac{\lambda}{3}\right)\tilde{f}_{11} - \tilde{g}_{11}, \quad \hat{h}_{51} = \left(1 - \frac{\lambda}{3}\right)\hat{f}_{21} - \hat{g}_{21}$$

$$\tilde{h}_{51} = \left(1 - \frac{\lambda}{3}\right)\tilde{f}_{21} - \tilde{g}_{21}, \quad \hat{g}_{11}(z) = \int_{-1}^{z}(1-\lambda z^2)\hat{f}_{11}\,\mathrm{d}z, \quad \tilde{g}_{11}(z) = \int_{-1}^{z}(1-\lambda z^2)\tilde{f}_{11}\,\mathrm{d}z$$

$$\hat{g}_{21}(z) = \int_{-1}^{z} (1-\lambda z^2)\hat{f}_{21}\,dz, \quad \tilde{g}_{21}(z) = \int_{-1}^{z} (1-\lambda z^2)\tilde{f}_{21}\,dz, \quad \hat{f}_{11} = \int_{-1}^{z} 2\rho_1\omega\tilde{u}_{(0)}\,d\eta$$

$$\tilde{f}_{11}(z) = \int_{-1}^{z}\left[\left(\frac{\tilde{m}^2\tilde{Q}_{11}\gamma_\beta}{\gamma_\alpha}+\frac{\tilde{n}^2\tilde{Q}_{66}\gamma_\alpha}{\gamma_\beta}\right)\tilde{\varphi}_{11}+\tilde{m}\tilde{n}(\tilde{Q}_{12}+\tilde{Q}_{66})\tilde{\varphi}_{21}-\tilde{m}\left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x\gamma_\alpha}+\frac{\tilde{Q}_{12}}{R_y}\right)\tilde{\varphi}_{31}\right.$$
$$\left.-\frac{1}{R_x}\tilde{\tau}_{xz(0)}-\tilde{m}\tilde{c}_{13}\gamma_\beta\tilde{\sigma}_{z(0)}-\rho\omega^2\tilde{\varphi}_{11}\right]d\eta-\left(\frac{1}{R_x}+\frac{1}{R_y}\right)z\tilde{\tau}_{xz(0)}$$

$$\hat{f}_{21} = \int_{-1}^{z} 2\rho_1\omega\tilde{v}_{(0)}\,d\eta$$

$$\tilde{f}_{21}(z) = \int_{-1}^{z}\left[\tilde{m}\tilde{n}(\tilde{Q}_{12}+\tilde{Q}_{66})\tilde{\varphi}_{11}+\left(\frac{\tilde{m}^2\tilde{Q}_{66}\gamma_\beta}{\gamma_\alpha}+\frac{\tilde{n}^2\tilde{Q}_{22}\gamma_\alpha}{\gamma_\beta}\right)\tilde{\varphi}_{21}-\tilde{n}\left(\frac{\tilde{Q}_{12}}{R_x}+\frac{\tilde{Q}_{22}\gamma_\alpha}{R_y\gamma_\beta}\right)\tilde{\varphi}_{31}\right.$$
$$\left.-\frac{1}{R_y}\tilde{\tau}_{yz(0)}-\tilde{n}\tilde{c}_{23}\gamma_\alpha\tilde{\sigma}_{z(0)}-\rho_1\omega^2\tilde{\varphi}_{21}\right]d\eta-\left(\frac{1}{R_x}+\frac{1}{R_y}\right)z\tilde{\tau}_{yz(0)}$$

$$\hat{f}_{31} = \int_{-1}^{z}[2\rho_2\omega\tilde{w}_{(0)}-\tilde{m}\hat{f}_{11}-\tilde{n}\hat{f}_{21}]\,d\eta$$

$$\tilde{f}_{31}(z) = \left(\frac{1}{R_x}+\frac{1}{R_y}\right)z\tilde{\sigma}_{z(0)}+\int_{-1}^{z}\left\{-\left(\frac{\tilde{c}_{13}}{R_x}+\frac{\tilde{c}_{23}}{R_y}\right)\tilde{\sigma}_{z(0)}+\tilde{m}\left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x\gamma_\alpha}+\frac{\tilde{Q}_{12}}{R_y}\right)\tilde{\varphi}_{11}\right.$$
$$+\tilde{n}\left(\frac{\tilde{Q}_{12}}{R_x}+\frac{\tilde{Q}_{22}\gamma_\alpha}{R_y\gamma_\beta}\right)\tilde{\varphi}_{21}-\left(\frac{Q_{11}\gamma_\beta}{R_x^2\gamma_\alpha}+\frac{2Q_{12}}{R_xR_y}+\frac{\tilde{Q}_{22}\gamma_\alpha}{R_y^2\gamma_\beta}\right)\tilde{\varphi}_{31}-\tilde{m}\tilde{f}_{11}-\tilde{n}\tilde{f}_{21}$$
$$\left.-\left(\frac{\tilde{m}\eta}{R_y}\right)\tilde{\tau}_{xz(0)}-\left(\frac{\tilde{n}\eta}{R_x}\right)\tilde{\tau}_{yz(0)}-\rho_2\omega^2\tilde{\varphi}_{31}\right\}d\eta$$

$$\varphi_{11}(z) = -\frac{z}{R_x}\tilde{u}_{(0)}+\int_0^z\left[\frac{2}{R_x}\tilde{u}_{(0)}-\tilde{m}\tilde{\varphi}_{31}-\left(\frac{c_{44}Q}{c_{45}^2-c_{44}c_{55}}\right)\tilde{\tau}_{xz(0)}\right]d\eta$$

$$\tilde{\varphi}_{21}(z) = -\frac{z}{R_y}\tilde{v}_{(0)}+\int_0^z\left[\frac{2}{R_y}\tilde{v}_{(0)}-\tilde{n}\tilde{\varphi}_{31}-\left(\frac{c_{55}Q}{c_{45}^2-c_{44}c_{55}}\right)\tilde{\tau}_{yz(0)}\right]d\eta$$

$$\tilde{\varphi}_{31}(z) = \int_0^z\left[\left(\frac{\tilde{m}\tilde{c}_{13}}{\gamma_\alpha}\right)\tilde{u}_{(0)}+\left(\frac{\tilde{n}\tilde{c}_{23}}{\gamma_\beta}\right)\tilde{v}_{(0)}-\left(\frac{\tilde{c}_{13}}{\gamma_\alpha R_x}+\frac{\tilde{c}_{23}}{\gamma_\beta R_y}\right)\tilde{w}_{(0)}\right]d\eta. \tag{A24a–r}$$

At the ε^2-order level

$$\tilde{u}_{(1)} = U_1+z\Phi_{x(1)}-\frac{\lambda z^3}{3}[\Phi_{x(1)}+z\tilde{m}W_1]+\tilde{\varphi}_{11} \tag{A25}$$

$$\tilde{v}_{(1)} = V_1+z\Phi_{y(1)}-\frac{\lambda z^3}{3}[\Phi_{y(1)}+z\tilde{m}W_1]+\tilde{\varphi}_{21} \tag{A26}$$

$$\tilde{w}_{(1)} = W_1+\tilde{\varphi}_{31} \tag{A27}$$

$$\tilde{\sigma}_{x(1)} = -\frac{\tilde{Q}_{11}\tilde{m}}{\gamma_\alpha}U_1-\frac{\tilde{Q}_{12}\tilde{n}}{\gamma_\beta}V_1+\left(\frac{\tilde{Q}_{11}}{\gamma_\alpha R_x}+\frac{\tilde{Q}_{12}}{\gamma_\beta R_y}\right)W_1+\tilde{c}_{13}\tilde{\sigma}_{z(0)} \tag{A28}$$

$$\tilde{\sigma}_{y(1)} = -\frac{\tilde{Q}_{12}\tilde{m}}{\gamma_\alpha}U_1-\frac{\tilde{Q}_{22}\tilde{n}}{\gamma_\beta}V_1+\left(\frac{\tilde{Q}_{12}}{\gamma_\alpha R_x}+\frac{\tilde{Q}_{22}}{\gamma_\beta R_y}\right)W_1+\tilde{c}_{23}\tilde{\sigma}_{z(0)} \tag{A29}$$

$$\tilde{\tau}_{xy(1)} = \frac{\tilde{Q}_{66}\tilde{m}}{\gamma_\beta}U_1+\frac{\tilde{Q}_{66}\tilde{n}}{\gamma_\alpha}V_1 \tag{A30}$$

$$\tilde{\tau}_{xz(1)} = \tilde{f}_{11}-\lambda\hat{f}_{11}+\int_{-1}^{z}\left[\left(\frac{\tilde{m}^2\tilde{Q}_{11}\gamma_\beta}{\gamma_\alpha}+\frac{\tilde{n}^2\tilde{Q}_{66}\gamma_\alpha}{\gamma_\beta}\right)\tilde{u}_{(1)}+\tilde{m}\tilde{n}(\tilde{Q}_{12}+\tilde{Q}_{66})\tilde{v}_{(1)}\right.$$
$$\left.-\tilde{m}\left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x\gamma_\alpha}+\frac{\tilde{Q}_{12}}{R_y}\right)\tilde{w}_{(1)}-\rho_1\omega^2\tilde{u}_{(1)}\right]d\eta \tag{A31}$$

$$\tilde{\tau}_{yz(1)} = \tilde{f}_{21} - \mu\hat{f}_{21} + \int_{-1}^{z} \left[\tilde{m}\tilde{n}(\tilde{Q}_{12} + \tilde{Q}_{66})\tilde{u}_{(1)} + \left(\frac{\tilde{m}^2 Q_{66}\gamma_\beta}{\gamma_\alpha} + \frac{\tilde{n}^2 \tilde{Q}_{22}\gamma_\alpha}{\gamma_\beta} \right) \tilde{v}_{(1)} - \tilde{n}\left(\frac{\tilde{Q}_{12}}{R_x} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y\gamma_\beta} \right) \tilde{w}_{(1)} - \rho_1\omega^2\tilde{v}_{(1)} \right] d\eta \quad \text{(A32)}$$

$$\begin{aligned} \tilde{\sigma}_{z(1)} = & -\left(\frac{1}{R_x} + \frac{1}{R_y} \right) z\tilde{\sigma}_{z(0)} + \int_{-1}^{z} \left\{ \left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x^2\gamma_\alpha} + \frac{2\tilde{Q}_{12}}{R_x R_y} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y^2\gamma_\beta} \right) \tilde{w}_{(1)} \right. \\ & - \tilde{m}\left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x\gamma_\alpha} + \frac{\tilde{Q}_{12}}{R_y} \right) \tilde{u}_{(1)} - \tilde{n}\left(\frac{\tilde{Q}_{12}}{R_x} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y\gamma_\beta} \right) \tilde{v}_{(1)} + \tilde{m}\tilde{\tau}_{xz(1)} + \tilde{n}\tilde{\tau}_{yz(1)} \\ & \left. + \frac{\tilde{m}\eta}{R_y}\tilde{\tau}_{xz(0)} + \frac{\tilde{n}\eta}{R_x}\tilde{\tau}_{yz(0)} + \left(\frac{\tilde{c}_{13}}{R_x} + \frac{\tilde{c}_{23}}{R_y} \right) \tilde{\sigma}_{z(0)} - \rho_2\omega^2\tilde{w}_{(1)} - 2\rho_2\omega\mu\tilde{w}_{(0)} \right\} d\eta. \end{aligned} \quad \text{(A33)}$$